On the Traversing

of

Geometrical Figures

By

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Oxford
At the Clarendon Press
1905

Henry Frowde, M.A.
Publisher to the University of Oxford
London, Edinburgh
New York and Toronto

Meiner Frau

The writer thanks the Delegates of the Clarendon Press for undertaking the publication of this venture. He acknowledges also his indebtedness to Professor E. B. Elliott, Mr. E. H. Hayes, and Mr. J. E. Campbell for testing a few statements submitted to them: but for the contents of the book the writer himself is entirely responsible, with the exception of the theorem for which he has to thank Mr. P. K. Tollit.

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ADDENDUM TO § 75.

If in the procedure of pp. 136 and 139 the Cartesian equation to the parabola, $y^2 = 4\,ax$, is substituted for the equation $\tan \phi = \sin \theta$, then, the equation to a point (xy) being

$$(x + a) \tan \Phi + (a - x) \tan \Theta = y,$$

we get for the tangent at (xy)

$$y = \frac{2\,a}{\tan \Phi - \tan \Theta}.$$

Whence if m is the foot of the perpendicular from P to the axis, and n_1 the foot of the perpendicular from B_1 to the directrix

$$Pm \,.\, Tn_1 = XS^2.$$

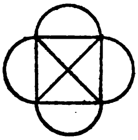
A similar method may of course be applied to the other focal conics.

On the continuous description or traversing of Geometrical Figures.

Introduction.

There is a puzzle of which, though it is probably a familiar one, the writer has never seen any treatment of any kind[1]. It is to describe or traverse continuously every line of a given figure without going over any line twice, the describing point being always kept on a line of the figure till the whole has been traversed.

The accompanying figure is an example.



The number of lines described will vary with the method chosen.

[1] After reading some parts of this investigation to the Oxford Mathematical Society, the writer was told by Professor E. B. Elliott that the negative theorem of § 5, viz. that a figure with more than two odd points could not be exhausted in one traverse, together with the reason there given, had been communicated to him in an oral tradition many years ago; but that the positive theorems were new to him.

Thus a traverse starting from the middle point of this figure may end with the description of only six lines of the figure, as in Fig. 1. That in Fig. 2 omits three lines of the figure. Fig. 3 omits two lines, and so does Fig. 4. Fig. 5 omits one line only.

Fig. 1.

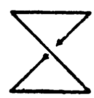


Fig. 2.

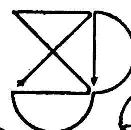


Fig. 3.

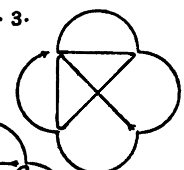


Fig. 4.

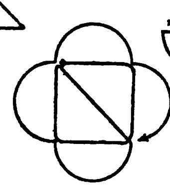
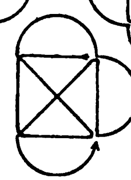


Fig. 5.



On trial it is always found that one line, at least, has been omitted. It will be proved that this must always be the case, the puzzle being really insoluble.

The object of the present investigation is to give a general theory of all such continuous traversing, whether the traverse completes every line of the figure or not.

The subject will be treated in two ways. In the first way, which may be called 'analytical,' complex figures will be assumed as given, and by traversing paths on them certain simple theorems about the number and nature of lines radiating from a point in the figure will be proved; the kinds of figure which can be traversed in a single traverse in the manner explained will be determined, and the number of separate traverses required in general for any figure. Finally, every figure will be resolved into a minimum number of elementary figures, each of which can be traversed in one traverse.

The second way, which may be called 'constructive,' starts with certain elementary figures which can be traversed in

one traverse, and by constructing complex figures out of them, arrives at the same results as the first way.

Some theorems will be used in both methods alike: and sometimes of two alternative forms of a theorem, both of which might be given in either method, one is given in Part I, and the other in Part II.

In Part III the principle of Duality is applied. This leads to a new kind of traverse, involving the ordinary conception of 'the point at infinity' upon a straight line: to extensions of certain kinds in the conception of the correspondence between two reciprocal figures: and lastly to new kinds of vector and coordinate.

Part I. Analytical Method.

§ 1. *Some definitions.*

A *figure*, for the present purpose, may be defined as a system of points joined by lines, such that every point in it is connected with every other point by a line or lines.

The expression *points of the figure* must be taken to exclude points from which only two lines radiate. For even if such lines make an angle with one another, they do not for the purposes of a mere traverse differ from those which make one continuous line, nor does the consideration of them enter into any of the problems of the first part of the present theory. 'Points of the figure' then is a name restricted to points from which an odd number of lines radiate, including the case where the point is the extremity of a single line, and to those points from which an even number of lines greater than two radiates.

A point from which an even number of lines radiates may be called an *even point*: a point from which an odd number radiates may be called an *odd point*. In this reference to points, the terms 'odd' and 'even' may be called the *denominations* of points.

A *line of the figure* is a line whose extremities are two adjacent 'points of the figure,' there being no other 'point of the figure' upon it.

A *path* consists of a number of adjacent 'lines of the figure,' such that one extremity of each of them coincides with an extremity of the one preceding it, and the other

extremity with the extremity of the one that follows it. A path from a point of the figure which returns to the same point is called a *circuit* through that point.

A circuit passing through only one point of the figure may be called a *loop*. At the point itself the branches of the loop count as two lines radiating from it.

To *traverse* in a figure, or in a part of it, is to trace a path along its lines, no line being traced twice over, ending at a point at which no path in the figure, or the given part of it, remains untraced.

The tracing of such a path may be called a *traverse.*

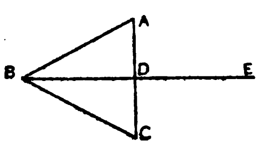
An *exhausted line* is one which has been traversed.

An *exhausted point* is one such that every 'line of the figure' radiating from it has been traversed.

A figure is said to be *exhausted by one traverse* when one single traverse exhausts all its points, or, what is the same thing, all its lines. Thus the problem of the puzzle is to exhaust the given figure in one traverse.

A point of the figure which is in the path of a given traverse is said to be an *included point* with reference to that traverse, and points of the figure not upon it may be called *omitted points.*

A point which is passed through but is neither the starting-point or end of a traverse may be called an *intermediate* point.



In the above figure D is an 'even point'; B and E are 'odd points.' A and C do not count as 'points of the figure.' The points of the figure are B, D, and E.

BAD and BCD count each as one 'line of the figure': the other 'lines of the figure' are BD and DE.

§ 2. *Some elementary properties of traverses.*

1. A traverse must exhaust the point at which it ends: for if any path from it were left untraversed, the traverse would leave the point by the path, and so it would not be the point at which the traverse ends.

2. At an 'intermediate point' the lines traversed must be even in number. For, since the process neither starts nor finishes there, a line traversed to it must have corresponding a line traversed from it, which may be called the 'conjugate' in that traverse to the former line.

3. If a traverse begins at a point A in a figure it must end at a point B which it exhausts. If we reverse the traverse and begin at B, retracing the same path, we shall not end at A unless A was exhausted by the first traverse. If it is not exhausted continue the traverse through A till it ends in some point C which it exhausts. The point B is evidently exhausted, and the traverse exhausts both its starting-point and its ending-point. Such a traverse as BAC may be said to be a *complete traverse*; for it comes to an end whether traced in the direction by which it arrives at any point, or in the opposite direction.

A 'complete traverse' then is one which exhausts both of its extremities.

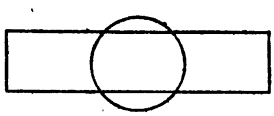
Any part, such as AB or AC, of a complete traverse, which ends at one of its extremities, may be called an *incomplete traverse*: and an 'incomplete traverse' exhausts only one of its own extremities.

4. If all the lines included in a given traverse are struck out of a figure, the 'denomination' of all intermediate points in the remaining figure is unaltered, but if the starting-point is not exhausted, its denomination is altered.

At any intermediate point an even number of lines has been traversed (§ 2, 2), therefore the lines which remain, when the traversed paths are struck out, are even in number if the point was even, and odd if it was odd. Thus the denomination of the intermediate points is unchanged. Supposing the initial point is not exhausted, if the traverse passes only once through it, the lines remaining when the traversed one is struck out are even in number if the point was odd, and odd if it was even. If the traverse returns to the initial point, it must leave it again after every return; for since the initial point is not exhausted, the traverse cannot end there. Consequently the traversed lines at it are odd in number. The remaining lines, therefore, when those traversed are struck out, must be even in number if the point was odd, and odd if it was even. Thus the denomination of the point is always changed.

§ 3. *Number of even points and of odd points in a figure.*

It is clear that a figure may have all its points even points: for a figure formed by the intersection of single circuits must have all its points even.



Also a figure may have all its points odd: for instance, if three radii of a circle are drawn, the centre is an odd point, and the points where the radii meet the circumference are odd points.

1. *If a figure has any odd points it must have more than one.*

Suppose all the points in a figure except the point A

are even points. Traverse the line AB leading from A to a next point B. Since B is an even point there must be another line from it conjugate to the path BA. Traverse this to a next point C: then similarly there must be from C a line conjugate to CB, and so on for every point reached, since they are all even points. If in this process we return to an even point already passed through, an even number of lines radiating from it have been traversed (§ 2, 2), therefore the untraversed paths remaining at it are even. Consequently the path by which we return to it must have a conjugate by which we pass away from it, and so for any subsequent return. Thus the process cannot end at any even point, and therefore must end at A.

If it ends at the first return to A, then A is exhausted (§ 2, 1). Thus the first path from it, and the line of return to it, make up all the paths at A, and therefore A is an even point.

If A is returned to more than once—loops being, of course, comprised in such return paths—before each successive return there will have been an odd number of lines traversed at it, and therefore the return path which ends the process will make up with these others an even number of lines at A. But, as A is exhausted, these must be all the lines at A. Therefore A is an even point.

Thus no figure, whether exhaustible or not by a single traverse, can have one odd point only.

If, therefore, a figure has odd points their number cannot be less than two.

2. *The number of odd points in a figure must be even.*

Let A be an odd point in a figure. Then, as before, a traverse from it cannot end in any even point. It cannot end in A, the starting-point, for then, as shewn

in the foregoing, A would be an even point. The traverse therefore ends in an odd point. Let this point be B. B then is exhausted (§ 2, 1).

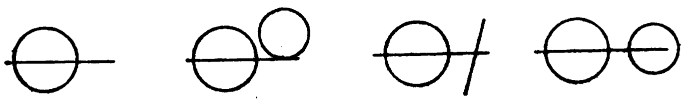
Let C be another odd point.

Strike out all the lines exhausted in the traverse from A. Then B is struck out, since it is exhausted. The point A, if not exhausted, becomes even, and the denomination of all other points is unchanged (§ 2, 4). There is, therefore, a figure left which contains one odd point C, or, if what remains consists of several separate figures, one of these has one odd point C. Hence, by the preceding, there must be another odd point, conjugate to C, in which a traverse begun from C must end. Thus in general it is clear that every odd point must have such a conjugate, and hence they occur in pairs always.

The above proof holds also if it is assumed that there are no even points.

The number of even points may be either even or odd.

First, if a figure has n even points, whatever may be the number of the odd points, one more even point can be added to it, by means of a loop which adds no odd points, or a line intersecting a line of the figure which may add two odd points also, or a circuit intersecting one line of the figure once only, where that is possible.



A figure with two or more odd points, whether it has even points or not, may be conveniently called an *odd-point figure.*

A figure which has even points only may be called an *even-point figure.*

§ 4. *Traverses of even-point figures.*

1. If a figure contains even points only, any traverse ends at the point from which it begins: and this point is exhausted.

This follows at once from the first part of the preceding section.

2. *A figure containing even points only can always be exhausted by one traverse.*

Supposing a given traverse T does not exhaust the figure. The points through which untraversed lines pass, are either included or omitted points. If there are any omitted points, since they cannot form a system unconnected with the rest of the figure, one at least must be such that a line passes from it to some included point, which line, therefore, must be untraversed. Thus, whether points are omitted or not, there will be an included point at which there is an untraversed line. Let A be such a point, and let an untraversed line pass from it to a next point B. B cannot be starting-point or end-point of T, for these two coincide (§ 4, 1) in one point which is exhausted (§ 2, 1). B then was either intermediate to the traverse T or omitted from it. If B was intermediate, the traversed lines at B are even in number (§ 2, 2). Therefore, since B is an even point, the untraversed lines at B are even in number. And if B is an omitted point, the untraversed lines are even in number. Hence the line BA must have an untraversed conjugate at B. Let this pass to C. Then, similarly, CB has an untraversed conjugate at C, and so on.

If the untraversed path returns to any of its previous points except A, it must leave them again by a conjugate line as before. Thus the untraversed path from A cannot

end at any of the other points. Hence the path must end at A itself, though it may return to A more than once.

This untraversed path then forms a circuit through A. But A is a point included in T; consequently when we come to A in this traverse, we can traverse the omitted circuit or circuits through A, and continue T from A as before.

Similarly all other omitted lines will fall either into circuits through points included in T, or into circuits through points in these circuits, and so on. Now if two circuits meet or intersect, the traverse of the one can be included in, or made continuous with, the traverse of the other. For in traversing one, when we come to a point which it has in common with the other, we can traverse that other returning to the aforesaid point, and then complete what remains of the traverse of the first. The two, in fact, make one complex circuit through any common point. And similarly for any number of meeting or intersecting circuits. Hence the untraversed lines fall into untraversed circuits, whether complex or not, which pass through points in T.

Thus the figure can be exhausted in one traverse, for each circuit or complex circuit can be described when we come to the point in T through which it passes, from that point and back to it, the traverse T being continued beyond each such point as before.

The proof may also be stated otherwise.

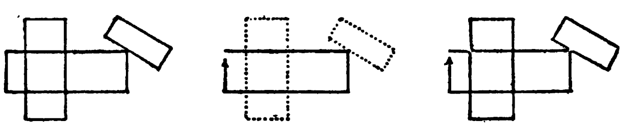
Suppose that at a point X there is an untraversed line. It may be shewn as above that a traverse starting from X along the line and confined to untraversed lines will end at X.

If now at any point Y there remains a line not included in either of the two previous traverses, it may be shewn that this forms part of a still untraversed circuit through Y.

Similarly if at a point Z there remains a line not included in any of the three previous traverses, this line forms part of a circuit not traversed in any of the previous traverses. Clearly the process may be continued till there remain no untraversed lines.

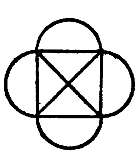
The whole figure then consists of a number of circuits, which must meet or intersect one another continuously, or the figure would be discontinuous. But if two circuits meet or intersect, the traverse of either can be included in that of the other, as above explained. Hence, obviously, the traverse of any one of the circuits of which the figure consists may be made to include, or be continuous with, the traverses of all the rest.

Thus the figure can be exhausted in one traverse, and consists, clearly, of one complex circuit. The following is a simple example:—



§ 5. *A figure with more than two odd points cannot be exhausted by one traverse.*

If there are more than two odd points, then one of them at least must be intermediate to any traverse. But the lines traversed at an intermediate point must be even in number (§ 2, 2). Hence one line at least at the inter-mediate odd point cannot be traversed, and the figure cannot be exhausted in a single traverse.



In the figure of the puzzle there are four odd points; hence it cannot be exhausted in a single traverse. It will be shewn hereafter that the figure can be so traversed that only one line is omitted.

§ 6. *A figure with two odd points only can be exhausted by one traverse.*

Suppose a figure has two odd points only.

If we start a traverse from an even point, one at least of the two odd points will be intermediate, and therefore the traverse cannot exhaust it.

We must therefore start a traverse T from one of the odd points. Let this be A, and let the other odd point be B. It follows from § 3, 2 that if a figure has two odd points only, any traverse which starts from one of them must end in the other.

T therefore ends at B, and exhausts it.

Suppose that T does not exhaust the figure. Strike out all the lines which belong to T. Then (§ 2, 4) all the points which remain are even points. The remaining lines thus form one even-point figure, or several such figures discontinuous with one another. In the former case the figure forms a circuit (§ 4), and must meet or intersect the path of T, since the original figure is continuous. It can therefore, as explained in § 4, be traversed in a traverse continuous with T, and the whole figure can be exhausted by one traverse.

In the latter case the several figures being discontinuous with one another must meet or intersect the path of T, or the original figure would be discontinuous.

Each then, being an even-point figure, forms a circuit through some point of T, and thus the traverse of each can be combined in one traverse with T (§ 4).

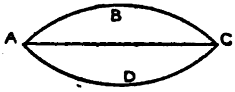
Hence the whole figure can be exhausted in one traverse.

§ 7. '*Single paths*,' '*Characteristics*' *and* '*Accessories*,' '*Single Circuits*.'

Let T be a complete traverse between two odd points in any odd-point figure. If we join its two extremities by a path P, which is not a circuit and nowhere returns on itself so as to form circuits, it follows from § 6 that the remainder of the paths of the traverse consist of circuits which meet or intersect the path P. The latter may be called a *single path* in distinction from one which, returning to its starting-point or to some other point or points in itself, is either a circuit or a single path combined with a circuit or circuits through points in it.

Such 'single path' of T may be called the *characteristic* of T.

The whole path of T, however, may be resolved into a single path with circuits in different ways. Thus this figure which can be exhausted in one traverse, may be resolved into the single path AC with circuit $ABCDA$, through A or C; or into the single path ABC with circuit ADC, &c.



The circuits in any resolution may be called *accessories* of the 'characteristic' of T in that resolution; and such as intersect no other part of the figure than this characteristic may be called its *peculiar accessories*.

According to § 4, 2, every even-point figure is such that it can be resolved into a number of circuits inter-

secting or meeting one another: and also that it forms one continuous circuit through any point in the figure.

In the case then of an even-point figure, a traverse T has properly no 'characteristic,' or 'single path,' or we may say its characteristic is a point (cf. § 40 fin.). But a circuit which returns only to its starting-point, in one traverse of it, and to no other point in itself may be called a *single circuit* of which all other circuits in the system are 'accessories,' and the whole system may be called a *complex circuit.*

§ 8. *A figure with two odd points can be reduced to a single path with accessories.*

It is evident from § 6 that a figure with two odd points only either consists of a single path, exclusively, between the odd points, or of a single path intersected or met by circuits; that is to say, of a single path and its accessories.

This may be also shewn as follows:—

A figure which can be exhausted by one traverse from one odd point to the other, must form a continuous path from one of them to the other. If the path returns from the second to the first point, the first branch and the return branch form a circuit at either of the two odd points, and there must be another branch by which it returns to the second point. And if the path returns oftener to the first point there will be always an odd number of branches connecting the two points. If there be $2n+1$ of these, $2n$ of them will form n circuits, at either point (or one complex circuit). The remaining one is not a circuit, but it may return to points in itself other than the starting-point. At such points circuits are formed,

and so clearly it can be resolved into a single path with accessories.

The branches of the n circuits may similarly have accessories, and the n complex circuits thus formed are accessories to the single path.

Thus the whole reduces to a single path and its accessories.

It will be shewn hereafter (§ 29) that a figure containing $2n$ odd points may be resolved into n single paths, with their accessories.

§ 9. *Rules for traversing figures which can be exhausted by a single traverse.*

The rules for the traversing of figures which can be exhausted by a single traverse may be stated in the terminology of the preceding section, thus:—

In a figure with two odd points only, traverse any path, single or otherwise, from one odd point to the other.

Mark mutually exclusive circuits through each point included in the path chosen, leaving no circuits unmarked.

Traverse the path first chosen again, stopping at each point to describe the circuits through it which have not been traversed already from some previous point, and continuing the original traverse between the points which belong to the original path until the other odd point is reached.

If the first traverse is a single path, that will be the characteristic of the whole traverse chosen.

The rule is the same for an even-point system, if we substitute for the characteristic single path, any circuit, single or otherwise, through the starting-point, which may be any point.

§ 10. *General types of figures with two odd points.*

In a figure some lines may be parts of a characteristic only, as OB (fig. on p. 21). Some may be parts of an accessory circuit or of a characteristic: thus OA_3 may be part of the characteristic BOA_3C, or if BOC is taken as a characteristic, OA_3 may be part of the circuit OA_3A_4.

Some lines again may be such that they cannot be parts of a characteristic. These are lines which in a complete traverse can only be traversed as parts of a circuit. These circuits may be conveniently called *fixed circuits*, since they must always be traversed as circuits. Such circuits, whether single or complex, are those which are only connected at one point in themselves with the rest of the figure. Thus a 'fixed circuit' may in a given resolution of the figure meet a characteristic in one point only, and any accessory through the same point in one point only, and no other part of the figure. Or the single point at which it is connected with the figure may have only accessories through it. Thus in Fig. 1 in the complex circuit $AHBCDMLBDKA$, which can only be traversed as a circuit, the line BD, e.g., cannot be part of a characteristic. EAF is the characteristic.

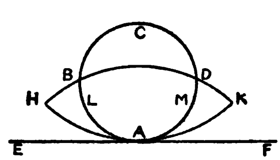


Fig. 1.

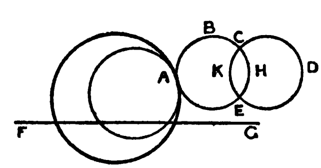
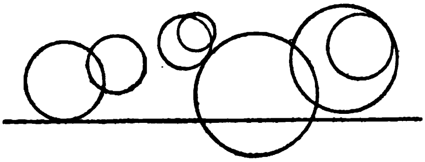


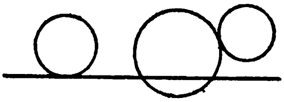
Fig. 2.

In Fig. 2 $ABCDEKHEA$ is a fixed circuit, connected with the rest of the figure by meeting an accessory to a given characteristic FG in one point A only, not in the characteristic: and meeting any other accessory through A also in A only.

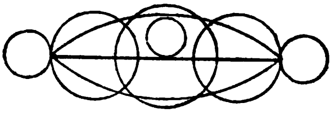
A loop is obviously a case of a fixed circuit. The following figure shews each kind of circuit and line: and



assuming the accessories may have any position, may be taken as type of any figure with two odd points. If we represent any circuit, whether simple or complex (except the complex in which the single circuits have but one point in common with one another), by a circle, the type may be abbreviated to this.



Since circuits may be through either odd point, or pass through both or neither, the following may be given as general type of a figure with two odd points: it being understood that any number of the circuits may be removed, while the straight line represents the essential part, the characteristic.



§ 11. *Definition of 'single lines.'*

Suppose $2n$ lines through a point are associated pair by pair in mutually exclusive circuits, i. e. circuits with no common segment: the maximum number of circuits which is possible in the most favourable case is obviously n. The

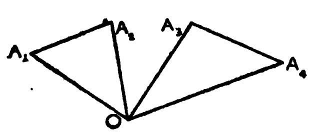
lines, however, may sometimes be paired in different ways to form circuits.

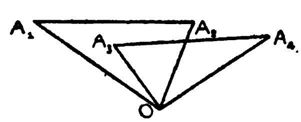
If AB is a 'line of the figure' joining two adjacent 'points of the figure' A and B, any path from B may be conveniently called a ***prolongation*** of the line AB considered as one of the lines radiating from A, and any path from A a ***prolongation*** of the line BA considered as one of the lines radiating from B.

Let the pair of lines OA_1 and OA_2 through O belong to a circuit A_1OA_2, and the pair OA_3 and OA_4 to a circuit A_3OA_4.

First, let these circuits meet only in O, and let no other line connect any part of either circuit with the other. Then OA_1 can only be paired with OA_2 to form a circuit, and OA_3 only with OA_4, and no arrangement will produce any other pairs.

Secondly, suppose the circuits intersect. Then, clearly, in whatever order we take the lines through O in pairs, we get each time two mutually exclusive circuits, and the number of the circuits is not altered.

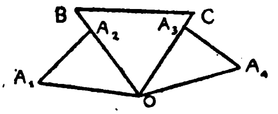


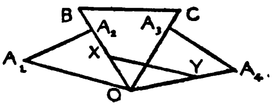


Thirdly, suppose the circuits do not intersect, but some path meets a line in each circuit, or its prolongation, not at O.

Let, e.g., BC meet OA_2 in B, and OA_3 in C. If we do not combine the lines in pairs consecutively, in the order in which they meet the path $A_1A_2BCA_3A_4$, but combine, e.g., OA_1 with OA_3C; OA_1 and OA_3 fall into the circuit OA_1A_2BCAO, and OA_2 and OA_4 cannot be combined in a circuit unless they are intersected by some other line, e.g., XY; but this case reduces to the

last, that of the intersecting circuits $O A_1 A_2 B C A_3 O$ and $O X Y$ and therefore may be neglected.





There are two lines then which do not enter into circuits and one circuit is lost. Thus the maximum is only obtained by combining the lines through O in consecutive pairs. The same follows whichever parts of the two circuits $B C$ connects.

In the other two cases the same procedure gives the maximum result, though in the second the order is indifferent.

And, in general, the maximum, when all the $2n$ lines can be combined in mutually exclusive circuits, is obtained by taking consecutively, in the order in which they meet a path which they all meet, the lines which can be combined in a circuit, whatever association is made of the figures of the three cases.

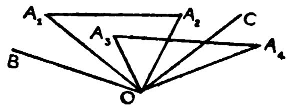
The figure, however, may be such that the maximum number of pairs, which can be combined in circuits, is less than n, whether the whole number of lines is $2n$ or $2n+1$: and if the maximum number of such pairs is m, the $2m$ lines of one resolution into m pairs are not necessarily the same as the $2m$ lines of another similar resolution.

All the mutually exclusive circuits through a point, each including one pair only of lines through it, form one complex circuit: but when these circuits are the maximum number, the maximum number of lines form parts of such a circuit. Thus the resolution may also be described as one in which a maximum number of lines are parts of a complex circuit through the given point.

Circuits which contain only one pair of lines through O may be called *component* circuits.

If at any point, when the lines through that point have been associated in the maximum number of component circuits possible for them, there remain lines which do not belong to these circuits, these lines may be called *single lines*.

Thus, in this figure, OB and OC are single lines.

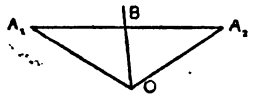


A resolution of the lines at a point which yields a maximum number of mutually exclusive circuits may be called a *maximum resolution*.

§ 12. *The number of 'single lines' at a point is constant, but their position may vary.*

Different resolutions may yield a different number of lines which do not form parts of circuits. But it is obvious that for every 'maximum resolution' the number of such lines is the same: for in every maximum resolution the number of component circuits at the given point is the same, and therefore the number of lines at the point belonging to component circuits is the same. Thus the number of lines which remain over, and are therefore single lines, must be the same in every maximum resolution.

If a single line OB, or a 'prolongation' of it, intersects a component circuit $A_1 OA_2$, then this part of the system at O may also be resolved into a circuit into which OA_1 and OB enter, so that OA_2 is the single line; or into



a circuit into which OA_2 and OB enter, so that OA_1 is the single line.

The remaining circuits will be unaffected, and the system will have either OA_1 or OA_2 or OB as one of its single lines. Similarly if OB, or a prolongation of it, is met by a line which also meets a circuit, or system of circuits connected with one another otherwise than at O, some other lines may in turn take the place of OB as single line. If OB can enter into a circuit with a line OA_1 of the system, some line must meet both OA_1 and OB, or prolongations of them, at some other point than O. Conversely if OB is not thus connected with any circuit through O, OB cannot enter into any circuit, and so always remains a single line. Such a line may be called a *fixed* single line.

As to variable single lines, it may be shewn that if there is more than one single line at a point, no two of them are interchangeable, in the above manner, with the same group of lines at that point. Let there be a system of component circuits at O, each connected with each otherwise than through O. This involves, clearly, that from any point, except O of course, on any one of them there is to any point on any other a path not passing through O. Let a single line OB_1 be connected with this system in the same way. Then there cannot be another single line similarly connected with the system at O. For, if possible, let OB_2 be such a line. Then OB_1 and OB_2, or prolongations of them, must be met by a line or path which meets each in some point other than O. If this path has no segment in common with a line necessary to make one of the component circuits, B_1OB_2 belongs to a new component circuit, and there is no single line. If this path has a segment in common with a line of a component circuit A_1OA_2, then OB_1 forms part of a circuit

with OA_1 or OA_2, and OB_2 with OA_2 or OA_1, and as the circuit A_1OA_2 has no segment of it common with any other circuit of the resolution, these two new circuits which replace it form with the remainder a system of mutually exclusive component circuits, and there are no single lines.

If the path which meets OB_1 and OB_2 has a segment or segments in common with two of the component circuits of the resolution, OB_1 and an adjacent line through O will belong to a circuit. OB_2 and an adjacent line will belong to a similar circuit, and with a part of the path which meets OB_1 and OB_2 the two remaining lines of the original circuit will form a circuit. These three circuits will be mutually exclusive, and will be exclusive of the remaining circuits. Thus again, there will be no single lines and the number of circuits will be increased: which is impossible as it is a maximum.

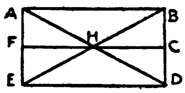
The same will hold, however many of the original circuits have a common segment with the path which meets OB_1 and OB_2.

Thus two single lines cannot be both connected otherwise than at O with one of the circuits through O, nor with any system of such circuits connected with each other otherwise than at O.

But the lines which can in different maximum resolutions become single lines in place of a given single line, must belong to groups with which it is connected otherwise than at O. Hence two single lines cannot be each thus interchangeable with the same group of lines.

It may also be shewn that if a single line is connected in the manner described with a system of circuits similarly connected, any of the lines belonging to the system at O may in turn be single lines, and from this the preceding result can be easily derived.

This result can also be got by the method of § 14. Cf. § 16 init.



In this figure, if AHB and EHD are taken as circuits through H, there remain no circuits into which FH and CH can enter. But these are not single lines at H because the maximum of circuits has not been taken. If we take AHB, CHD, and HEF as circuits, there are no single lines. H therefore is not 'a point which has a single line.' At the points B and C, on the other hand, every resolution leaves a single line, B and C are therefore said to have single lines.

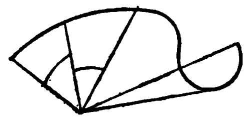
§ 13. *Every odd point has at least one single line.*

Every odd point has at least one single line. For the maximum of possible circuits is when the lines in mutually exclusive pairs belong to circuits, one circuit for each pair. Hence at least one line must remain in any resolution which does not belong to any circuit.

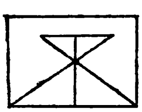
If there are any single lines at an even point, it is clear that their number must be even.

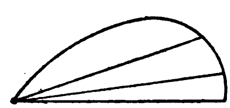
§ 14. *Point at which there cannot be more than one single line.*

If a path meets each line at O, or a prolongation of it at some other point than O, then if O is an even point it has no single line, and if odd it has one and no more.

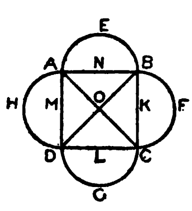


For if we take the lines in consecutive mutually exclusive pairs in the order in which they, or their prolongations, meet this path, each pair, or their prolongations, form with a part of the given path a component circuit. Hence if the number is even, $2n$, there are n circuits mutually exclusive, and no single line. And if the number of lines through the point is odd, there is one single line and no more. Thus, if there is more than one single line, there can be no path which meets all lines through it, nor can the prolongation of any line through it meet all the rest.





In the figure of the puzzle every line from any point in it except O meets the single circuit $AEBFCGDH$ twice, and, e. g., the single circuit $ANBKLM$ twice. The point O is within both these circuits, and every line from it meets them.



Thus A, B, C and D being odd points have each one single line. And O being an even point has no single line.

§ 15. *The number of single lines at a point is not a function of the number of odd points.*

The number of single lines at any point does not depend upon the number of odd points in the figure.

If from a point within a single circuit we draw lines to the circuit, the point can have only one single line (§ 14), however great an odd number of lines we draw; but the number of the other odd points is the same as the number of the lines.

§ 16. *Nature of the figure at points at which there is more than one single line.*

If at a given point O there is more than one single line L_1, L_2, &c., it follows from § 12 that each of them is either unconnected with any other part of the figure except at the given point O, or is connected with a system of circuits connected with one another otherwise than at O, which system is unconnected with the corresponding system for any other single line except at the point O.

This result may also be deduced from § 14.

For suppose L_1 is connected otherwise than at O with a system S_1 of component circuits through O connected with each other otherwise than at O, and connected with no other part of the figure except at O.

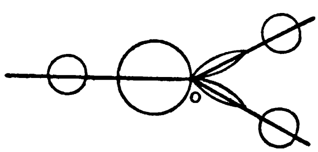
The number of lines of S_1 at O is obviously even.

Suppose L_2 is similarly connected with the system S_1. Then since from every point to every other point in the system of lines comprising S_1 and L_1 and L_2, there is a path not passing through O, the point O is on the periphery of a closed figure; two of the lines at O form part of this periphery, and the remainder, or their prolongations, meet

the rest of the periphery, each at some other point than O. Therefore by § 14 it follows that since the number of lines at O belonging to the system composed of S_1 and L_1 and L_2 is even, there is no single line in this system. But the system cannot be affected by the resolutions of any other part of the figure at O, since these parts are not connected with it except at O. Thus L_1 and L_2 cannot be single lines.

L_2 then cannot be connected with S_1 except at O.

Similarly, if S_2 is the system corresponding to L_2, L_1 cannot be connected with S_2 except at O. Again, S_2 cannot be connected with S_1 except at O, for S_1 and S_2 would form one system, and L_1 would be connected with S_2 otherwise than at O. Also L_1 and L_2 cannot be met by another line except at O, for they would then become part of a circuit and not single lines.



If therefore there are m single lines at O the figure at O consists of m separate figures which have no connexion with each other except at O, but all have O as common point.

§ 17. *Maximum of single lines at a point.*

If there are several single lines at a point O, it is evident that in the separate figure corresponding to each such single line there must be at least one odd point beside the point O counted as in that figure only.

Consequently if O is an odd point, and there are $2m_1+1$ single lines at O and therefore $2m_1+1$ such separate figures, the whole number of odd points in the whole figure is not less than $2m_1+2$.

If O is an even point, and there are $2m_1$ single lines, there are $2m_1$ corresponding figures, and at least $2m_1$ odd points.

Thus if O is an odd point the number of odd points in the whole figure is at least one greater than the number of single lines at O, and if O is an even point, the number of odd points in the whole figure is not less than the number of single lines at O.

It follows that the number of single lines at an odd point must be at least one less than the number of odd points. Thus in a figure with two odd points, an odd point can only have one single line. Which is obvious also otherwise, from the resolution of such a figure into a characteristic and accessories.

Again, it follows that the maximum number of single lines at an even point cannot be greater than $2n$, the number of odd points. This formula holds for an even-point figure. For if we put $n=0$ this would give zero for the number of single lines at a point in an even-point figure, which is correct. (See also § 42.)

Again, the maximum number of single lines at an even point in a figure with two odd points is two: which again is verifiable from the resolution of the figure into a characteristic and accessories.

Thus, by § 5, if at any odd point there is more than one single line, and at any even point more than two, the figure cannot be exhausted by one traverse. [Also it will appear from § 28 that if the number of single lines at any point is m, and m is even, the number of traverses which can exhaust the figure is not less than $\frac{m}{2}$: if m is odd not less than $\frac{m+1}{2}$.]

It is evident that when the number of odd lines at an even point is a maximum in proportion to the whole

number of odd points, every odd point has but one single line: and if the number of single lines at an odd point is such a maximum every other odd point has only one single line.

§ 18. *An odd point which can be exhausted by a single traverse must have one single line and no more, and conversely.*

At an odd point there must be one single line at least. If there is only one, then the remaining lines at the point form a system of mutually exclusive circuits. Thus a traverse along a single line to the point exhausts the circuits as well as the single line. Hence the point is exhausted.

Conversely, suppose a traverse which exhausts an odd point arrives at it by a single line—since there must be at least one single line. If the point is not exhausted, then there remains an even number of lines at it. Hence the traverse leaving by one of them must return by another, and these two form a circuit through the point.

Similarly the remaining pairs fall into circuits.

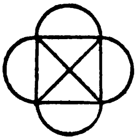
Thus the lines at the point form one single line and a set of circuits.

Hence if any odd point in a figure has more than one single line it cannot be exhausted in one traverse. The point B_1 in Fig. 1, § 26, is an example.

§ 19. *Any point in the puzzle figure can be exhausted by a single traverse.*

It follows from § 14 and § 18 that an odd point, such that all lines through it or their prolongations meet a

path, must be exhaustible in a single traverse. Hence it follows that any odd point in the figure of the puzzle can be exhausted by a single traverse.



It follows also by § 14 that an even point within a closed circuit, all lines through which meet the circuit, can be exhausted by one traverse only: for the lines through it form a continuous complex circuit.

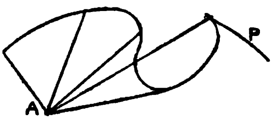
Hence the middle point of the puzzle figure can be exhausted by a single traverse.

§ 20. *Two odd points which can be exhausted by one traverse.*

If the lines at an odd point are related to a path in the manner described in § 14, clearly any line through it can be taken as its single line, the remainder falling into mutually exclusive circuits.

Suppose there are two odd points A and B each thus related to a line, not necessarily the same line. Let P be any path to A. P meets the closed figure at A formed as described in § 16.

Fig. 1.



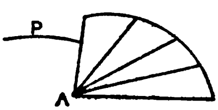
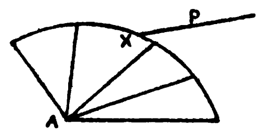


Fig. 2.

If P is a prolongation of one of the lines at A (Fig. 1), or meets one of the lines at A which form part of the periphery of the closed figure (Fig. 2), then this line may

be taken as single line at A, and a traverse along it continuous with P exhausts also the circuits into which the rest of the lines at A fall.

Fig. 3.

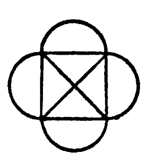


Next let the point X where P meets the periphery fall between two points in which lines from A, or their prolońgations, meet the periphery (Fig. 3). Let a traverse continuous with P pass from X along the periphery and along that one of the aforesaid lines from A which has an even number of lines from A on each side of it. Then the lines on each side of the last line fall into component circuits which have no common segment with the traverse. Thus the traverse exhausts A.

Hence a traverse along any path to A can exhaust it. The same is true of B. Therefore if A and B are joined by any single path, this forms a characteristic of a traverse which, with its accessories, exhausts both of them.

Two odd points then of this kind in a figure can be exhausted by the same traverse. And the number of traverses between them will be the same as the number of single paths which can join them.

It follows from this last section that any pair of corner points in the puzzle figure, which are the odd points, can be exhausted by one traverse.



§ 21. *General condition that two odd points should be exhaustible by the same traverse.*

Suppose two odd points are exhausted by the same traverse. Then the characteristic of this traverse evidently terminates at each point in a line which is a single line at that point. Consequently two odd points can be exhausted by the same traverse, if some line which can be a single line at the one and some line which can be a single line at the other are connected by a continuous path which has no common segment with any line which must be part of a circuit through either point in the corresponding resolutions.

§ 22. *Positive condition that two odd points should not be exhaustible by the same traverse.*

Let A be an odd point with one single line L_1.

If there is any circuit single or complex with which L_1 is connected otherwise than at A (§ 16), let this be S_1.

Let S_2 be a circuit not connected with S_1 or L_1 except at A. Then S_2 remains a circuit in any resolution.

S_1 and L_1 form a closed figure.

Suppose a path P to A meets this figure, but not S_2. A traverse continuous with P exhausts the lines at A which belong to the closed figure (§ 20) and exhausts the circuit S_2. Thus this traverse exhausts A.

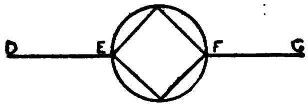
If, however, P meets S_2 at any point except A, if a traverse continuous with P exhausts the lines at A which belong to S_2, it cannot exhaust the lines at A which belong to the aforesaid closed figure, and conversely.

It follows that between points which can only be connected by help of what must be part of a circuit through one or

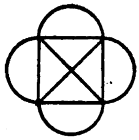
other of them in any maximum resolution at the point, there cannot be a traverse which exhausts both of them.

And if two odd points, each with one single line, are not so connected, they can be exhausted by one traverse.

In the figure E and F cannot be exhausted by the same traverse, nor D and F, nor E and G.



In the case of the puzzle, any pair of odd points can be exhausted by the same traverse, for each is obviously joined to any other by a line which can be a single line at both.



§ 23. *In any figure two odd points can always be found which can be exhausted by the same traverse.*

If we start a traverse at any odd point it must end at an odd point and exhaust it. This latter point then is exhaustible by one traverse, and therefore there must be only one single line at it in any resolution.

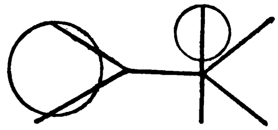
Take any of the lines at it which can be single lines, then a traverse along this line starting from the point obviously exhausts it, but the traverse must end in some other odd point and exhaust it. Thus one and the same traverse exhausts both points.

Hence there are at least two points which can be exhausted by the same traverse.

§ 24. *Relation of the number of odd points in a figure to that of the whole number of single lines in a given resolution.*

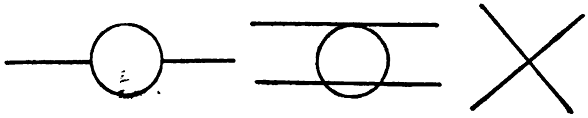
A single path joining two odd points, with no other odd point upon it, may be called a 'branch.' A branch clearly terminates in single lines. Two odd points cannot be joined by two branches, for these would form a circuit, and would not terminate in single lines. Two branches must have no common part.

If then all the odd points are connected by branches with one another so that the branches are connected by the coincidence of extremities,



it is evident that if the number of branches is μ and the number of odd points $2n$, $\mu = 2n-1$, $n = \frac{\mu+1}{2}$.

If, however, each branch is connected with another, either by circuits only, or by the coincidence of points in them other than extremities,



to each pair of odd points will correspond only one branch, and therefore $\mu = n$ and the number of odd points $= 2\mu$.

[Hence if μ is odd the minimum number of odd points is $\mu+1$, and therefore (as will appear from § 28) the number of traverses necessary is not less than $\frac{\mu+1}{2}$.

If μ is even the minimum number of odd points is $\mu+2$, and the minimum number of traverses $\frac{\mu+2}{2}$. In either case the maximum possible is 2μ odd points and μ traverses.

The whole number of single lines in the figure $=2\mu$, each termination of a branch counting as a single line at the corresponding extremity.

Thus if M is the whole number of single lines in the figure, if $\frac{M}{2}$ is even the number of necessary traverses is not less than $\frac{\frac{M}{2}+2}{2}$, and if $\frac{M}{2}$ is odd the minimum is $\frac{\frac{M}{2}+1}{2}$. The maximum is $\frac{M}{2}$.]

§ 25. *Maximum number of odd points in an odd-point figure, at each of which there is more than one single line.*

It follows from the preceding that if $2n$ is the number of odd points in a figure, the maximum number of branches is $2n-1$, and therefore the maximum number of single lines is $4n-2$.

Since each odd point has at least one single line and there are $2n$ odd points, $(4n-2)-2n, = 2n-2$, is the maximum number of single lines which could be distributed among the odd points which are to have more than one single line, in addition to the one included already in the $2n$.

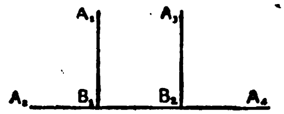
Corresponding to the total $2n-2$ there will be a maximum of odd points with more than one single line, if there can be assigned to each of them the minimum number of single lines at such points, which is three: that is if two single lines beside the one already assigned can be added to one

odd point after another until the total $2n-2$ is exhausted. Thus, as this total is $(n-1)2$, if the figure permits, $n-1$ odd points would receive two extra single lines each, and there would remain $n+1$ odd points with one single line each.

A superior limit then of the number of odd points, each with more than one single line, is $n-1$, and an inferior limit of odd points with but one single line is $n+1$.

It may now be shewn that these are respectively the true maximum and minimum.

By the preceding, if the maximum of odd points with more than one single line can be $n-1$, each point has three single lines.



Two such points B_1 and B_2 must be connected by a path.

Let this be a single path B_1 B_2 which terminates in a single line B_1 B_2 at B_1, and in a single line B_2 B_1 at B_2. Let the other single lines at B_1 and B_2 respectively be B_1 A_1, B_1 A_2, and B_2 A_3, B_2 A_4. Then there are six odd points; of these two have three single lines each, and four have one single line only. Thus there is the required minimum of the one kind and maximum of the other kind, since in this case $n=3$.

Similarly if any number of points, at each of which there are three single lines, are joined in the same way,

if the whole number of odd points is $2n$, there are $n-1$ points with three single lines each, and $n+1$ with but one single line each.

Thus $n+1$ is the true minimum of points with but one single line in a figure of $2n$ odd points, and $n-1$ the maximum of odd points with more than one single line.

§ 26. *Minimum number of pairs of odd points each of which can be exhausted by the same traverse.*

If A is an odd point at which there is one single line, a traverse beginning along this line must end at some other odd point B and exhaust it, so that there can be but one single line at B. A and B may be called conjugates of one another.

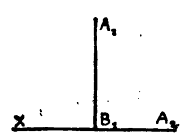
Let A_1 be a point at which there is but one single line. There must be another point X which is conjugate to A_1 and has also only one single line.

Let there be another point A_2, with but one single line, and let its conjugate be Y.

Let P_1 be a single path for the traverse from A_1 to X, and P_2 be a single path for the traverse from A_2 to Y.

Suppose that Y, the conjugate of A_2, coincides with one of the first pair, with X for example.

Fig. 1.



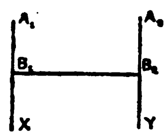
P_1 and P_2 must terminate in a common segment, $B_1\ X$, at X; otherwise there would be two single lines at X (Fig. 1).

Then clearly A_2 is conjugate to A_1, and there are three pairs of conjugates $A_1\ X$, $A_2\ X$, and $A_1\ A_2$. Also since odd points occur in pairs the addition of A_2 to the first pair necessitates another odd point, which is B_1 in the figure.

Suppose X and Y do not coincide.

First let X and Y be joined by a single path. As before, this must have a common segment at X with P_1 and another at Y with P_2. And it is easily seen that every pair of the four points A_1, A_2, X, and Y is a pair of conjugates.

Fig. 2.



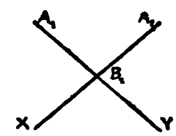


Fig. 3.

Hence there will be a smaller number of pairs of conjugates if X and Y are not conjugate and therefore connected only by means of a circuit in any resolution.

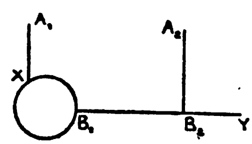
Let a fixed circuit pass through both (Fig. 4).

Fig. 4.

Then there are three pairs of conjugates only, $A_1 X$, $A_2 Y$, $A_1 A_2$, and four odd points.

If a fixed circuit passes through one of them only, as in Fig. 5,

Fig. 5.



there are six pairs of conjugates, $A_1 X$, $A_2 Y$, $A_1 A_2$, $A_1 Y$, $A_2 B_1$, $B_1 Y$. Thus when to the first pair $A_1 X$, another point A_2 is added, the minimum of pairs of conjugates which the three points A_1, X, A_2 necessitate in proportion to the whole number of odd points which they necessitate, is given in the cases represented in Figs. 1 and 4.

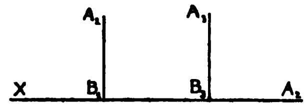
In Fig. 1, A_2 involves with A_1 and its conjugate the minimum of pairs of conjugates, if A_1 has the same conjugate

X with A_2; the new pairs due to A_2 being A_1 A_2 and A_1 X.

Suppose now another point A_3 with but one single line added to Fig. 1.

By the preceding A_3 involves with A_1 a minimum of pairs of conjugates if they have the same conjugate X, and the new pairs are A_1 A_3 and A_3 X. Similarly A_3 involves with A_2 a minimum if A_3 and A_1 have the same conjugate X, and the corresponding pairs are A_2 A_3, A_3 X.

Fig. 6.

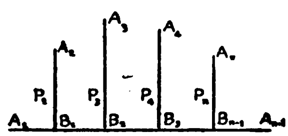


There will then be three new pairs A_1 A_3, A_2 A_3, and A_3 X.

This will happen if the single path P_3, from A_3 to X, has a common segment both with P_1 and P_2 terminating in X, as in Fig. 6.

The new odd point necessitated by A_3 is B_3, the point where P_3 meets P_2. This case coincides with that of Fig. 3, and so the latter comes really to the same as the case where two more points are added to the first pair, A_1 X, on the same principle as in Fig. 1.

Fig. 7.



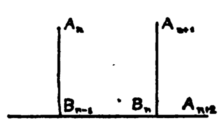
If points with one single line are added successively in the same way (Fig. 7), it is clear that every successive point will involve with the rest a minimum of pairs of conjugates in proportion to the whole number of odd points necessitated by the given number of points with but one single line. Now we cannot add to the figure any

more odd points with more than one single line each without adding more points with only one single line. For in this figure, if the number of odd points is $2n$, the number of those which have more than one single line is $n-1$, and by the theorem of the preceding section, this is the maximum possible of such points. Hence the arrangement in the figure gives the minimum of pairs of conjugates in proportion to the whole number of odd points in a figure. And since every point in it at which there is but one single line is conjugate to every other, the whole number of pairs of conjugates is $\dfrac{\lfloor n+1}{\lfloor 2 \lfloor n-1} = \dfrac{n(n+1)}{2}$. This therefore is the required minimum number.

This conclusion may be got without using the theorem of the preceding section.

It is clear that one more odd point with more than one single line must be added to the figure by a path P_{n+1} having one of its extremities B_n upon one of the preceding paths. The other extremity cannot be on any other path but P_{n+1} (Fig. 8),

Fig. 8.



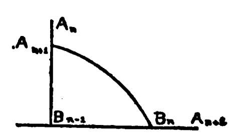


Fig. 9.

for otherwise a circuit would be formed, and A_{n+1} and B_n would be points with but one single line, as well as one at least of the points which had previously more than one single line.

Thus A_{n+1} must be a point with but one single line, and it is evident that we cannot add a point with more than one single line, without adding a corresponding point with only one single line.

Hence, as before, the figure gives the required minimum of pairs of conjugate odd points.

We get also thus an independent demonstration of the theorem of the preceding section; for it is evident that if we begin with a single pair of conjugates (and there must be at least one such), every point with more than one single line which we can add necessitates a corresponding point with but one single line. Thus, if there are $n-1$ such additions to the first pair, there will be a maximum of points with more than one single line, and their number will be $n-1$, while the number of odd points with but one single line will be $n+1$.

In Fig. 4, A_2 involves with A_1 a minimum of pairs of conjugates, and as all the odd points are points at each of which there is but one single line, there is a maximum of such points.

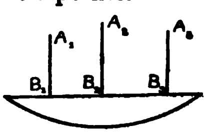


Fig. 10.

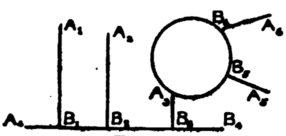


Fig. 11.

If the figure is continued in the same way (Fig. 10), each new point will involve the minimum number of pairs of conjugates, which number will be $\frac{n(n+1)}{2}$ out of a total of $2n$ odd points: and all the points being points with but one single line each, there is at the same time a maximum of points with but one single line. Thus though the only points which can be conjugate to one another are those with but one single line, the number of pairs of conjugates does not depend on the number of this kind of odd points; for it may be a minimum when they are either at a maximum or at a minimum. And it may be a minimum when their number is neither a maximum nor a minimum, as is easily shewn by a combination of Figs. 7 and 10, as in Fig. 11.

It will be observed that the expression for the minimum

number of pairs of conjugates shews (what has appeared in the case of Fig. 7) that in a figure with the minimum number of points with but one single line, viz. $n+1$, every such point is conjugate to every other.

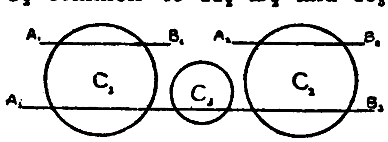
§ 27. *Omission from a figure of a path with accessories joining two odd points.*

Let A and B be odd points in a figure with $2n$ odd points, n being > 1. Strike out any single path P joining them with all its peculiar accessories. Then if A and B remain they become even points, and the remaining points have their denomination unchanged. $2n-2$ odd points therefore remain, and the remaining lines form one or more figures, such that every figure contains one or more pairs of odd points. For if there were a figure containing even points alone, it must form a circuit, and as it is not connected with the rest of the figure it must be connected with P, and must therefore be a peculiar accessory of P, whereas, by hypothesis, all such have been struck out. The remaining figure or figures then must all contain odd points, and the odd points occur in them in pairs.

The same is true if we strike out all the accessories of P.

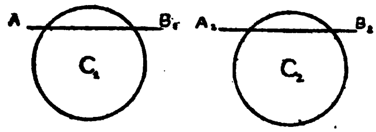
It follows that in a figure with four odd points, the remaining lines form one figure with two odd points.

Thus in the figure there being six odd points A_1, B_1, A_2, B_2, A_3, B_3; C_1 being an accessory common to $A_1\ B_1$ and $A_3\ B_3$, C_2 common to $A_2\ B_2$ and $A_3\ B_3$, and C_3



peculiar to $A_3\ B_3$: if we strike out $A_3\ B_3$ with its peculiar

accessory there remain two figures each with two odd points, viz. the figure formed by A_1 B_1 and its accessory C_1, and that formed by A_2 B_2 and its accessory C_2.



§ 28. *The number of traverses required to exhaust a figure of* 2n *odd points is* n.

Suppose a figure has $2n$ odd points. Strike out any single path joining any two odd points with its peculiar accessories.

Then by the preceding there remains a figure or figures containing $2n-2$ odd points; and each such figure contains at least one pair of odd points.

In any of these remaining figures strike out a path joining two odd points with its peculiar accessories. Then there remains a figure, or figures containing $2n-4$ odd points, each figure containing at least two odd points. Continuing the process after $n-1$ paths are struck out there remains a figure containing two odd points. This can be exhausted in a single traverse. Thus the whole figure can be exhausted in n traverses. And since no traverse can exhaust, or change to even points, more than two odd points, it is evident that n is the minimum number.

§ 29. *Resolution of a figure into a minimum of figures traversable in one traverse.*

It is clear from this that every figure with $2n$ odd points can be resolved into n different paths, each from

one odd point to a second: that is into n separate figures, each of two odd points, and each therefore traversable in one traverse, associated with one another, no two of such paths or figures having any line, or part of a line, in common.

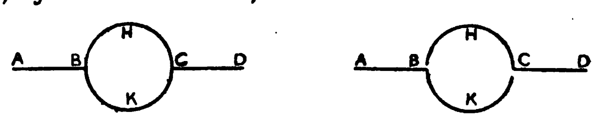
If the figure can remain continuous after the omission of each successive traverse, however the odd points are associated in pairs, the number of ways in which it can be resolved into n such components, is at least the same as the number of ways in which $2n$ points can be arranged in n pairs, and that is

$$\frac{|\underline{2n-1}}{2^{n-1}\,|\underline{n-1}}.$$

There may be a larger number, because it may happen that between a given pair of odd points there are several single paths which with their accessories form separate systems not coinciding with one another. If there are no accessories at all, the above expression is the whole number of ways in which the set of n figures can be obtained, on the given assumption of continuity.

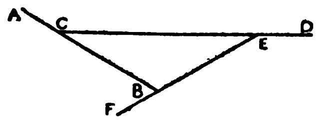
Instead of striking out a path between any two odd points, we might, in consequence of § 23, have each time struck out a traverse exhausting two odd points. In this second method the number of ways of resolution into n paths may be diminished, because in some figures not every pair of odd points can be exhausted by the same traverse.

Thus the figure $ABCD$, which has four odd points, may, by the first method, be



resolved into the figure AKC, and the figure BHD, among other resolutions; a resolution which cannot be given by the second method, because neither the pair of

odd points A and C, nor the pair B and D, can be exhausted by the same traverse. Similarly the figure $ABCDEF$, with six odd points,

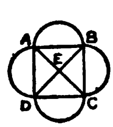


can be resolved by the first method into AB, CD, and EF; a resolution which cannot be given by the second method.

§ 30. *Maximum traverses.*

It is obvious from § 2, 3, that a traverse which exhausts a maximum of lines in a figure of more than two odd points must be a 'complete traverse.' In the case of a given figure the pairs of points which can be exhausted in a single traverse are determined by the theorems of §§ 21 and 22: and the nature of the particular figure must determine which of these traverses is a maximum.

If two points are exhaustible by the same traverse, the traverse can be performed by joining them by any single path, taking them as characteristic of the traverse, and describing this with its accessories as explained in § 9. There will be as many ways of describing the traverse as there are ways of joining the two points by a single path.



Thus in the puzzle figure, if we make the restriction, for simplicity, that AEC and DEB are always traversed on one line, so that such a path as AEB is excluded. There are fourteen characteristic paths for the traverse which

exhausts B and A, and for that which exhausts B and C. There are seventeen such paths for the traverse exhausting B and D. Thus under the given restriction there are altogether forty-five characteristic paths of traverses starting from B, or any odd point, which exhausts that point and another.

§ 31. *Figures in which the number of lines omitted in one traverse can have the minimum ratio to the whole number of odd points.*

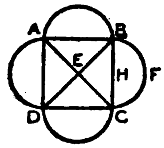
If a traverse is made which exhausts a pair of odd points in a figure, at each of the remaining odd points there must be at least one line not traversed.

If at one odd point one line only is lost, while this joins the point to another odd point, and is only one line of the figure, and if this is the only line lost at the other point, only one line is lost for the pair, which is the minimum loss for a pair of odd points.

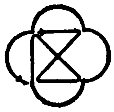
Suppose the $2n$ odd points of the figure can be arranged in n pairs, such that in at least $n-1$ of them each separate pair has its members joined by one line of the figure, while the suppression of none of these causes the remainder of the figure to become discontinuous. The figure can be traversed so that only those joining lines are lost, so that the loss is a minimum and the traverse a maximum.

For if we strike out these lines the odd points which they join become even or disappear, and the remaining points do not change their denomination. Thus the remaining figure, which is continuous, contains only two odd points, and can be exhausted in a single traverse. A traverse, then, exhausting these two odd points loses only the lines struck out, and is a maximum traverse.

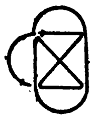
The puzzle figure is an instance in which the loss can be of the minimum kind. Each of the odd points in it is connected with each of two others by one line of the figure. Hence if we strike out one such line, the rest of the figure can be exhausted by a traverse from either of the other odd points.



Such traverse then omits only one line. If we omit *BHC*, for instance, the remainder can be completely described from *A* or *D*; for instance, thus:

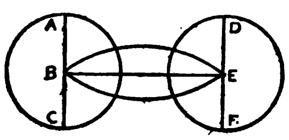


or for instance, thus: if we omit *BFC*.



If, however, we omit *BD* so that *A* and *C* are the remaining odd points, the corresponding traverse loses two lines of the figure, *BE* and *ED*.

The following figure with six odd points, *A*, *B*, *C*, *D*, *E* and *F*, can be described with the loss of two lines only, viz. *AB* and *DE*, or *AB* and *EF*, or *BC* and *DE*, which is the minimum loss for a figure with six odd points.



§ 32. *The maximum traverse of a figure with four odd points.*

The maximum traverse for a figure with four odd points may be thus determined.

Suppose a traverse made which exhausts two odd points. Strike out the lines of the traverse. Then by § 27 there remains a continuous figure with two odd points. This figure consists of a single path, with its peculiar accessories, connecting the two odd points: for any accessories not peculiar to it will have been included in the traverse, and therefore removed.

Thus any 'complete traverse' of the figure will leave a single path between two odd points with its peculiar accessories.

Find the pair of odd points for which such path is a minimum: then clearly the maximum traverse is that which exhausts the two remaining odd points. Thus in Fig. 1, BHC and BFC are minimum single paths. Hence

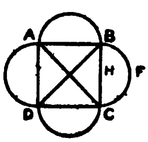


Fig. 1.

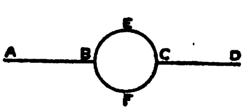


Fig. 2.

a traverse exhausting A and D is a maximum and leaves only one line, BHC or BFC untraversed. Similarly in Fig. 2 the maximum traverse omits BEC or BFC.

It follows from § 30 that under the restriction there made, there are, from any odd point, twenty-eight different characteristic paths for traverses which exhaust every line of the figure but one; and, of course, there are many more if this restriction be removed.

Part II. Constructive Method.

§ 33. *A figure which can be exhausted in one traverse must either have all its points even, or only two of them odd.*

The results of §§ 8 and 29 suggest another way of treating the subject.

A figure which can be exhausted in one traverse forms one continuous path from the starting-point to a different one, or back to the same point. These are the only two cases.

In the first case all points of the figure except the first and last must be produced by the return of the path upon itself: and thus all points in the figure are even points, except the first and last, which are odd points.

Since, however, the path may intersect itself at the first and last points, there may be more than one line at these points: and as each return obviously causes an even number of branches, the whole number of lines at these points remains odd.

In the second case, as the first and last points coincide, and all intermediate points must, as in the first case, be even, every point of the figure must be even.

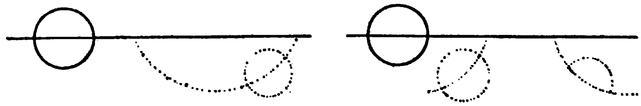
Consequently if a figure can be exhausted in one traverse, it must either have all its points even, or it has two odd points and no more.

§ 34. *The construction of figures from 'elementary paths.'*

Let the paths described in the previous section be called ***elementary paths***; those with two odd points ***elementary odd paths,*** and the others ***elementary even paths.*** It is evident that every figure is either an elementary path or can be produced by the association of two or more such paths whether of the same or of different kinds. The association must be such that any two elementary paths intersect or meet, or are intersected or met by another path, whether elementary or not.

No two of the elementary paths of which a figure is composed are to have a common segment.

If two paths are put together so as to have a common segment, this must be considered to belong to one of them only. The remainder of the figure will be one path of which the extremities of the common segments are the extremities, if the corresponding original path was a circuit; or else two paths, if the original path was not a circuit, one of which begins at one end of the common segment and the other at the other.



§ 35. *An even-point figure can be exhausted in a single traverse.*

It is clear that in the association of figures, whether elementary paths or not, an odd point in one figure can only become even by being made to coincide with an odd point in the other figure.

Hence a figure which has only even points cannot arise out of the association of a figure with odd points and a figure with even points only; but either out of the association of figures each of which has all its points even, or of figures each of which has odd points.

In the first case, the figures associated are circuits.

If two circuits are associated they meet or intersect, and each of them is a circuit through a 'point of the figure' in the other figure, or through a point upon one of its lines. Hence each can be included in a traverse of the other, and they form one circuit which can be exhausted in one traverse.

And similarly if any number of such figures are associated.

In the second case where the figure is produced by the successive association of elementary odd paths, we may omit the species where an extremity of one such path coincides with one extremity of the other, the other extremities not coinciding. For clearly two paths so associated form together one elementary odd path between their remaining extremities. The only species which remains is the association of two odd paths such that each extremity of the one coincides with one of the other. These form a circuit and can be exhausted therefore by one traverse. Succeeding odd paths then can only be added in pairs, each pair forming a circuit like the first two. Thus the figure is always formed of circuits which meet or intersect: and each new circuit being obviously included in the previous traverse, the whole figure can be exhausted by one traverse.

Hence if a figure with even points only be formed by the association of elementary paths with odd points it can be exhausted by one traverse. And the same has been shewn of those formed from elementary paths which

are circuits. Therefore a figure containing even points only can always be exhausted in one traverse and constitutes a circuit.

§ 36. *A figure with two odd points only can be exhausted by one traverse.*

Suppose a figure has two odd points only. If it is an elementary path it can be exhausted in one traverse. If it is not, then it can be resolved into such (§ 34).

A circuit will not count as a component, for it would obviously be regarded as a part of any part of the path it intersected, inasmuch as it could be traversed by the same traverse as that part.

If two of its components, which must be elementary paths with odd points, had their extremities coinciding each with each, they obviously form a circuit and can be excluded as before.

If two such elementary paths have one odd point coinciding, they clearly form a path, with two odd points, exhaustible in one traverse, and so count only as one elementary path.

The only case therefore which remains is the association of single elementary paths which have none of their odd points coinciding. The association of two such, produces a figure with four odd points, and that of more than two, a figure with more than four odd points. Thus the association of elementary odd paths must produce a figure with more odd points than the given path has.

Hence there can be no such resolution of the given path into components. Thus, it must itself be a single elementary path, and therefore can be exhausted in one traverse.

§ 37. *The number of odd points in an odd-point figure must be even.*

Of the elementary paths the one which is not a circuit has two odd points, the other, the circuit, has none.

If the elementary paths with odd points are associated, either the odd points of the two paths coincide, and therefore there are no odd points: or only one odd point of the one coincides with one of the other, so that two odd points become one even point, and there remain two odd points in the figure: or none of the odd points coincide, and so there are four of them. Thus in any case if there are any odd points their number is even.

Similarly, if another odd path is added, the number of odd points is either unaltered or increased by two, or diminished by two. Hence, in general, a figure arising from such paths must have an even number of odd points. The association with a path which is a circuit clearly makes no difference to the number of odd points. Hence the number of odd points in any odd-point figure is even.

§ 38. *Construction of figures from elementary odd paths.*

For the general construction of figures from elementary odd paths, we need only consider the case where no odd point of an elementary path coincides with one of another.

For if one odd point of one coincide with one of another the two become one elementary odd path.

If the odd points of one coincide with those of another, each with each, the two form an elementary even path. If one odd point of an elementary path P_1 coincides with one of an elementary path P_2, and the other odd point

of P_1 with one of an elementary path P_3, P_2 P_1 and P_3 constitute one elementary odd path.

From this again follows the theorem of the preceding section, viz. that the number of odd points in a figure must be even.

§ 39. *The number of traverses required to exhaust a figure of* 2 n *odd points is* n.

In the construction of a figure for the purposes of this theorem the association of elementary even paths may be neglected, for such a path adds no odd points to the figure; and as it is included in the traverse of any single path which it meets or intersects, it adds no traverse to those already required.

We need then only consider the association of elementary odd paths, and, by the previous section, of these again only those such that no odd point of one elementary path coincides with one odd point of another.

Suppose two such associated. If an odd point of the one is not on the other it remains an odd point, and if it is on the path of the other it still remains an odd point, since by hypothesis, it does not fall upon an odd point.

Thus as there are four odd points the figure cannot be exhausted in one traverse. It requires then at least two traverses; and it can be exhausted in two, that is, by the traverse of each elementary path.

Besides these traverses there will be others formed by taking any pair of points for the extremity of a traverse. For if we strike out a path between any two points with its accessories, then there remains a figure with two odd points (§ 27) which require only one traverse.

Add now another elementary path in the same way. The traverse of this cannot be included in any of the

preceding traverses, and therefore one more traverse will be required; and so for each such path added.

Thus it is evident that the number of traverses is the same as the number of pairs of odd points, that is n.

§ 40. *Interpretation of the zero value of* n *in the foregoing formula.*

The number of traverses for a figure with $2n$ odd points being n, if we put $n = 0$ we get zero for the number of traverses for a figure with no odd points, that is, for an even-point figure. The result is capable of interpretation, so that the formula may be of universal application. The traverses for which the formula was found are of single characteristic paths with their accessories: and the formula means that a figure of $2n$ odd points is resolvable into n single paths with their accessories, these latter not determining the number of such figures, for that is determined solely by the number of characteristic paths. The value zero then is correct for the figure with no odd points, for this contains no single path at all.

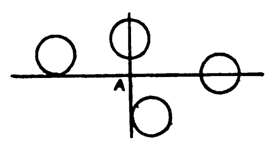
This is really the same as the statement that the characteristic of an even-point figure is a point (§ 7 fin.).

§ 41. *Maximum of single lines through a point.*

Since in an elementary odd path all points between the odd points are formed by the return of the path to points in itself, and are even points, at any even point upon a characteristic path there is at least one circuit. If there is only one circuit there are two single lines at the given point, which belong to the characteristic, and as the addition of circuits cannot increase the number of single lines, every even point on the characteristic has two single lines only. At an even point not on a

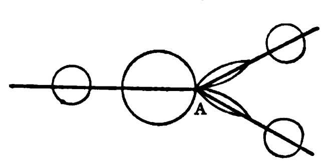
characteristic, there are only circuits. At the odd points there is but one single line. Thus the maximum of single lines at any point, whether 'point of the figure' or not, is two.

Let two elementary odd paths be associated, and let their characteristics intersect in a point A, which is not a 'point of the figure' on either path and at which therefore there are no circuits. This has four single lines, two on each path, provided the paths have no other point of intersection or meeting. This is evidently the maximum at a point, as giving the maximum for each of the paths upon which it is. Since the paths do not intersect or meet at any point but A, the figure will consist of four single paths radiating from A, none of the accessories of the one path intersecting those of the other.



If A is a point on either or both of the elementary paths, there are circuits at it. These will fall under the same condition if the maximum of single lines is maintained, and no circuit accessory to the four paths will meet any accessory to the other except at the point A.

Similarly if n elementary odd paths are associated, the maximum of single lines is reached when their characteristics all meet in the same point and form $2n$ single paths radiating from it with accessories, such that no accessory of one path meets an accessory of any other except at the point A.



Thus the maximum number of single lines possible at

such a point as A is the same as the number of odd points in these elementary paths, that is $2n$.

A is necessarily even and the whole number of odd points on the figure is $2n$.

Hence it follows that in any figure the number of single lines at an even point cannot be greater than the number of odd points.

Since an odd point can only have one single line in its own elementary path, the maximum of single lines for it, if two elementary odd paths are associated, is obtained when it coincides with a point which in the combined figure has two single lines in the other path. As before, the three paths radiating from the point common to the two original paths, can only have accessories which do not meet except at the given point. It is clear then that for n paths we get the maximum number of single lines at an odd point, if $n-1$ of them meet as in the previous figure, and an odd point of the remaining one coincides with the point of meeting, A; the $2n-1$ paths thus formed having only such accessories as do not meet the accessories of any other except at A. The number of odd points of the figure is $2n$ and the number of single lines at A is $2n-1$.

Thus the number of single lines at an odd point must be at least one less than the number of odd points in the figure.

§ 42. *Interpretation of zero values and of the minus sign.*

The formula for the number of single lines at an even point holds for the even-point figures, for if we put $n=0$ we get zero for the number of single lines at any point in an even-point figure. This corresponds to the fact that if we take an odd-point figure with the minimum number

of odd points, that is two, which is an elementary odd path, and if this has at some even point A two single lines, when we bring the extremities together, the figure becomes an even-point figure, and at A there are only circuits and no single lines.

If in the formula for the maximum number of single lines at an odd point, $2n-1$; we put $n=0$, we get -1 for the maximum of single lines at an odd point in an even-point figure.

Now in such a connexion the *minus* sign indicates impossibility, and this corresponds to the fact that it is impossible for an even-point figure to have an odd point.

The nature of this result is apparent when we reflect that if the value of the maximum had been zero this would have indicated that there could be no single lines at an odd point in the figure, and not that an odd point was impossible.

§ 43. *An odd point which can be exhausted by a single traverse must have one single line and no more, and conversely.*

If there is more than one single line at an odd point, there must be at least three, for the whole number of single lines at the point must be odd.

Suppose there are three, L_1, L_2, and L_3, at a point A. Then, by § 41, these are in figures F_1, F_2, and F_3 respectively which meet only at A, and have at A only circuits beside the corresponding single line. That is, A in F_1 is a single-line point, in F_2 it is a single-line point, and in F_3 it is a single-line point. A traverse along L_1 to A exhausts the circuits there belonging to F_1, it may then exhaust the circuits belonging to F_2 and F_3 at A; it may then proceed along L_2 and exhaust L_2, but it cannot then

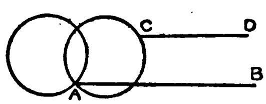
return to A since all the circuits at A are exhausted. Hence it cannot exhaust L_3.

Thus if A is exhaustible in a single traverse there can only be one single line at A.

And clearly if there is only one single line at A, the point can be exhausted in a single traverse.

§ 44. *Condition that two odd points should be exhausted by the same traverse.*

Suppose a traverse T exhausts a point A. If it starts from A it will leave A by a single line AB after exhausting the circuits through A into which the remaining lines at A must fall.



If T exhausts another odd point E, then E must be on an untraversed path from A, which must be a prolongation of AB. Hence points, such as C, upon a circuit through A belonging to the resolution of which AB is the single line, and not on any other path from A, or points, such as D, only connected with A by means of such a circuit, cannot be exhausted by T, for these circuits have all been traversed.

If therefore there are lines which in any maximum resolution must be parts of a circuit through A, no odd point which is only connected with A by means of such a line can be exhausted by the same traverse as A.

Conversely, it is evident that if two odd points, each of which has one single line, can be connected by a path which is the prolongation of a single line at each and has no segment common with any line which is necessarily

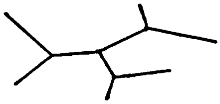
a part of one of the component circuits at either of them, they are exhaustible by the same traverse. For clearly one traverse exhausts the given path and the component circuits at each of its extremities.

§ 45. *In any figure two points can always be found which are exhaustible by the same traverse.*

In the elementary odd paths of which a figure is composed, two points, of course, in each can be exhausted by one traverse before the combination.

But in the combination an odd point of one path may fall on a point in another path such that in the figure there is more than one single line at the point, and then it cannot be exhausted by one traverse: or two odd points belonging to different paths may be connected only by means of lines which must in any maximum resolution be parts of a circuit at one or both of them.

But all the odd points cannot become such as to have more than one single line: if that were so, since no two such single lines could be intersected by one and the same line of the figure, the single lines would form a system of branches without any limit, and the figure would not be finite.



Thus if there are odd points with more than one single line, branches (single lines) which radiate from each such point, and are more than two in number, each begin paths terminating in a single line at an odd point which has only one single line. Let P_1, P_2, and P_3 be such paths radiating from a point A and ending in the single-line odd

points B_1, B_2, and B_3. Then clearly the same traverse will exhaust any pair of the terminating odd points.

It is also easy to shew that not every pair of odd points must be such that one of the points is only connected with the other by means of a line which must belong to a circuit at one or other of them in any maximum resolution.

For if the odd points of a path P_1 are A_1 and B_1, and B_1 is only connected with a path P_2, which has odd points A_2 and B_2, by a fixed circuit (for instance)

B_1 is only connected with A_2 and B_2 by means of the circuit, but B_1 and A_1 are necessarily connected otherwise than by this circuit or any at A_1.

The further interposition of fixed circuits in the direction of a traverse from B_1 through A_1, cannot prevent the traverse from passing to an odd point which it can exhaust; for an odd point must obviously always come before the circuit, in the direction of the traverse.

A point on the traverse such as A_1 can only be rendered not exhaustible by the traverse through the presence of more than one single line at the point, and thus this case reduces to the former.

The subject is treated more generally in § 47.

§ 46. *The minimum number of points in an odd-point figure, at each of which there is only one single line.*

As explained in § 38 we have only to consider that association of elementary odd paths in which no odd point of any path coincides with an odd point of another.

Let P_1 be an elementary odd path, with odd points A_1 and

B_1. Then either no odd point of P_1 falls upon another path; or one odd point only of P_1 falls upon another path; or both odd points of P_1 fall on another path P_2; or one odd point of P_1 falls upon a path P_2 and another odd point upon another path P_3.

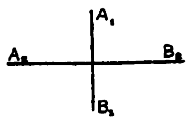


Fig. 1.

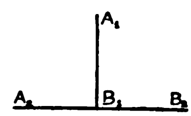


Fig. 2.

If two paths P_1 and P_2 are associated, and no odd point of the one is upon the other, every odd point of the figure has one single line only (Fig. 1).

If B_1 coincides with a point of P_2 through which there are no circuits, there are three single lines at B_1, and three odd points of the figure each have one single line only (Fig. 2).

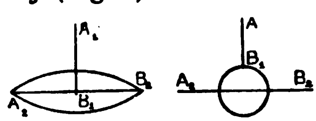


Fig. 3.

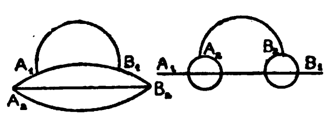


Fig. 4.

If B_1 is on a circuit belonging to P_2, there is one single line at B_1, and every odd point of the figure has one single line only (Fig. 3). If A_1 and B_1 both coincide with points in P_2 through which there are circuits, then again every odd point of the figure has one single line only (Fig. 4).

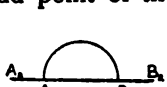


Fig. 5.

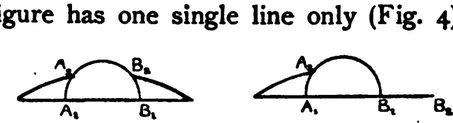


Fig. 6.

If A_1 and B_1 coincide with points in P_2 through which there is no circuit, P_1 with the part of P_2 between A_1 and B_1 forms a circuit, and thus at each odd point of the figure there is but one single line (Fig. 5). If A_1 and B_1

are on P_2, and either A_2 or B_2, or both, upon P_1, circuits are formed, so that at each odd point of the figure there is one single line only (Fig. 6).

In these figures and in the remaining ones formed by varying the position of the odd points of P_2, the minimum number of odd points which have each but one single line is given by Fig. 2.

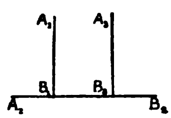


Fig. 7.

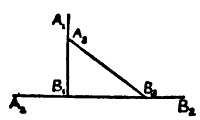


Fig. 8.

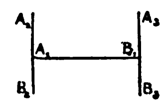


Fig. 9.

If with the two paths P_1 and P_2, associated as in Fig. 2, we associate in the same manner another elementary odd path P_3, so that B_3 is upon one of the other odd paths, and A_3 on no other, we add one more odd point with only one single line, and for the three united paths there are four odd points which have one single line only (Fig. 7). If in the last case we bring A_1 or A_2 or both upon another path, circuits are formed and all the odd points have one single line each (Fig. 8).

Suppose now, to take the last case of the division with which we started, one odd point, A_1, of a path P_1 is on a second path P_2, and the other, B_1, on a third path P_3, as in Fig. 9. Let there be no circuits at A_1 and B_1, and let no odd point of P_2 and P_3 fall on any other path.

Then $A_2\,A_1$ and $A_3\,B_1$ may be considered two elementary odd paths, each of which has one odd point, coinciding with a point upon the elementary odd path $B_2\,A_1\,B_1\,B_3$. Thus the figure reduces to Fig. 7; and if odd points of P_2 or P_3 are brought upon any other of the paths, such modifications belong to the type of Fig. 8.

We have therefore only to consider the method in which each successive elementary odd path is associated with the

preceding ones, by having one of its odd points upon a point in one of the preceding paths at which there is no circuit. Fig. 7 then gives the minimum of odd points each with one single line. Clearly the minimum will be maintained if we go on adding new paths in the same way (Fig. 10). As we begin with two odd points which have each a single line, and every successive path adds one, it is evident that the minimum number for n paths, that is for a figure with $2n$ odd points, is $n+1$.

Fig. 10.

§ 47. *The minimum number of pairs of points each of which can be exhausted by the same traverse.*

If two elementary odd paths meet one another, it can be easily shewn by a consideration of the various possible figures, that in each of the following three figures there is a minimum of pairs, viz. three, each of which can be exhausted by the same traverse.

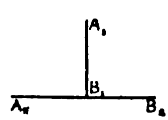


Fig. 1.

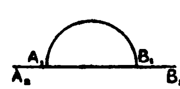


Fig. 2.

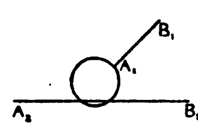


Fig. 3.

If in a figure two elementary odd paths, P_1 and P_2, do not meet, they must be met by some other path X. A consideration of the possible cases easily shews that the minimum of points in P_1 and P_2, which can be combined in pairs each exhaustible by the same traverse, is given when an odd point of each either coincides with a point at X at which there is no circuit, or is connected with X only by a circuit, as in Figs. 4, 5, and 6.

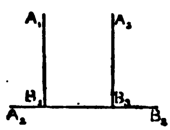


Fig. 4.

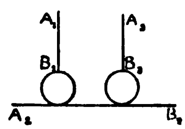


Fig. 5.

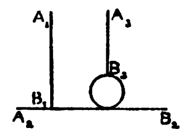


Fig. 6.

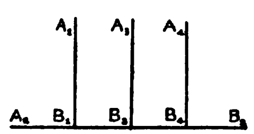
In these minimum cases one point only in P_1 is combinable with one point only in P_2, and P_1 and P_2 have between them only one pair of points which can be exhausted by one traverse.

Now clearly a figure has a minimum number of pairs each of which can be exhausted by the same traverse, when every two of its component elementary odd paths have between them a minimum of pairs of the required kind.

Hence there is a minimum when any two component paths which meet have between them only three such pairs, and any two which do not meet have between them only one such pair.

If then, for example, we associate paths P_3, P_4, &c., with a path P_2 in the same way as the path P_1 is associated with P_2 in Fig. 1, which gives figures of the type of Fig. 4,

Fig. 7.



every path satisfies the above conditions. And as one odd point only in each of the paths P_1 P_3 P_4, &c., can be paired with one odd point only in each of the rest, and with two odd points in P_2, and the odd points in each of the paths P_1 P_3 P_4, &c., cannot be paired with one another, while those in P_2 can, it is easy to see that the whole number of pairs is

$$n+(n-1)+(n-2)\dots 3+2+1=\frac{n(n+1)}{2}$$

Similarly if we add to Fig. 3, for example, by placing one odd point of each component path upon the same circuit, the condition of a minimum, stated above, is also clearly satisfied.

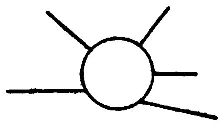


Fig. 8.

In this case one odd point only in each path can be paired with one odd point only in every other, and the two odd points in each path can be paired with one another. The whole number of pairs is therefore evidently

$$n+(n-1)+(n-2)\ldots 3+2+1=\frac{n(n+1)}{2}$$

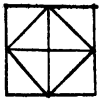
as before.

The theorem of § 45 is a particular case of the theorem of this section.

Part III. Application of the Principle of Duality.

§ 48. *Reciprocal figures. Notation.*

For simplicity only those figures will now be considered the lines of which are either straight or composed of straight lines[1]. Thus if more than one line of the figure joins two points of the figure one of these may be a straight line, and the rest must be rectilinear contours. For instance, the figure of the puzzle may be drawn in this way:—



Every angular point will now be accounted a 'point of the figure,' which will obviously make no difference to the preceding theorems, and the expression 'angular points of the figure' will be substituted for 'points of the figure,' and every straight line comprised between two angular points of the figure will be accounted a 'line of the figure.'

The reciprocal of a figure has a straight line corresponding to each angular point of the reciprocated figure, and an angular point corresponding to each of its lines. The reciprocal is drawn on any principle which secures that, if a point A is on the line corresponding to a point B, the point B is on the line which corresponds to the point A. If this is so, it follows obviously that to the line joining A and B must correspond the intersection of the lines which corre-

[1] For curvilinear figures see § 57, p. 88.

spond to A and B. This principle also secures that to collinearity in either figure will correspond concurrence in the other.

The following notation will be used.

A represents a point in a figure and (A) represents the corresponding line in the reciprocal figure. AB represents a line passing through the points A and B, and (AB) represents the point in the reciprocal figure which corresponds to the line AB. From the preceding it follows that (AB) is the point on which (A) and (B) intersect.

$A^{-}B$ represents the finite line comprised between the points A and B: similarly $(AB)^{-}(CD)$ represents the finite line comprised between the points (AB) and (CD).

$BA^{\wedge}AC$ represents the angle BAC comprised between the straight lines BA and AC: similarly $(A)^{\wedge}(B)$ represents one of the angles comprised between the two lines (A) and (B).

The kind of traverse which we have been considering will be called a *linear traverse*, in distinction from another kind to be presently described.

The sign $\backsimeq$ will be used for correspondence. Thus $A \backsimeq (A)$ means that (A) is the line corresponding to the point A, and $AB \backsimeq (AB)$ means that (AB) is the point corresponding to the line AB.

§ 49. *Relation between the linear traverses of reciprocal figures.*

For the purposes of a linear traverse the lines of the reciprocal figure are those which correspond to the points of the reciprocated figure, and are terminated by points which correspond to lines of the reciprocated figure.

It follows, therefore, that if any two straight lines in the reciprocal figure intersect, their point of intersection

will not count as a point of the figure, unless there corresponds to it a line in the reciprocated figure.

If any point in a figure consisting of finite lines, has only one line radiating from it, the corresponding line in the reciprocal figure will be of indeterminate length, and have only one point fixed upon it. For the present purpose such figures may be excluded; or we may suppose such a line to be of any arbitrary finite length, and to be on only one side of the point fixed in it.

It may be shewn that the reciprocal of a figure of $2n$ odd points need not itself have $2n$ odd points; that the reciprocal of a figure which can be exhausted by one linear traverse cannot always be exhausted by one such traverse, and conversely that if a figure cannot be so exhausted its reciprocal sometimes may be.

A single circuit always reciprocates into a circuit, but this is not necessary for a complex circuit.

A complex circuit consists of a single circuit with accessory single circuits (§ 7). Consider then the complex circuit formed by the triangle ACB and the triangle DEF, the point D being upon CB and between C and B.

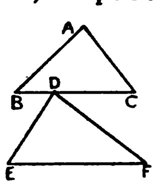


Fig. 1.

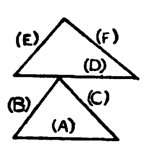


Fig. 2.

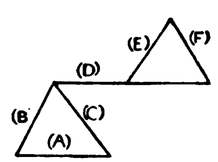


Fig. 3.

The reciprocal may be of the type in Fig. 2, and therefore also a complex circuit. But though the line (D) must pass through the point (BC), this point need not fall upon $(DE)\ (DF)$, but may fall outside it as in the type of Fig. 3. See the figure in § 53. This figure is not a circuit, but it can be exhausted by one traverse, for it has only two odd points, (BC) and (DE).

Next let the triangles meet at one angular point only, as the triangles ABC and CDE.

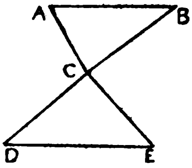


Fig. 4.

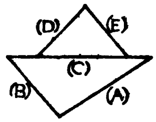


Fig. 5.

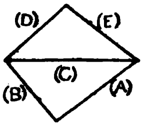


Fig. 6.

The reciprocal of Fig. 4 may be of the kind shewn in Fig. 5.

If AC and CE are in the same straight line, the points (AC) and (CE) coincide, and (DC) and (CB) coincide if the lines DC and CB are in the same straight line, as in Fig. 6.

Here again the reciprocal is not a circuit, but being a figure with two odd points can be exhausted by one traverse.

Next let one side, AB, of one triangle ABC be intersected by the sides of the other DEF in the points X and Y.

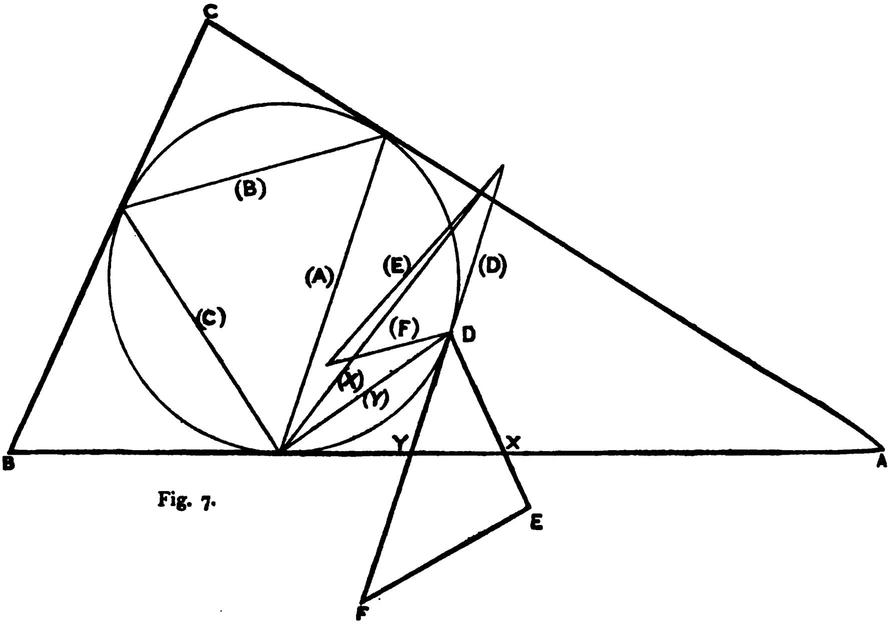


Fig. 7.

The reciprocals of the two triangles need not have intersecting sides, as is shewn in the figure, where poles and polars are taken with respect to the circle inscribed in ABC. And in any case the reciprocal figure has two odd points, the intersection of the lines (D), (X) and (E), and the intersection of the lines (D), (F) and (Y).

It may easily be shewn in general that two rectilinear single circuits, associated to form one complex circuit, may reciprocate sometimes into a circuit, and sometimes into a figure with one or more pairs of odd points.

If a single circuit has several accessories, more than one of these may be such as to give rise to a separate pair of odd points in the reciprocal figure; as, for instance, accessories of the kind shewn in Figures 5 and 7.

Conversely a figure with $2n$ odd points may be such that, whatever the value of n, its reciprocal may be a figure which can be exhausted in one traverse, for this reciprocal may be a complex circuit.

A figure with two odd points may reciprocate into one which also has two odd points.

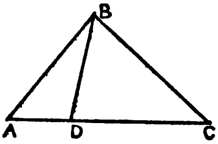


Fig. 8.

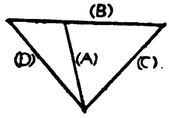


Fig. 9.

For instance, Fig. 8 reciprocates into the type of Fig. 9:

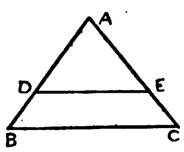


Fig. 10.

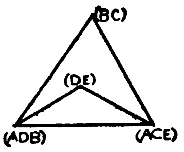


Fig. 11.

Fig. 10 into the type of Fig. 11.

In general the reciprocal of a circuit may be a circuit, or it may be a figure with one or more pairs of odd points.

Also, as is obvious from the above, the reciprocal of a figure with $2n$ odd points, may be a figure with a number of odd points greater than $2n$, or less than $2n$, or with no odd points at all.

§ 50. *The reciprocal of a linear traverse is an angular traverse.*

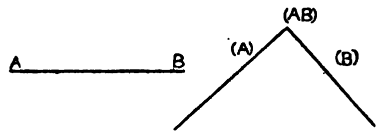
It is evident from the preceding that theorems which are true for the linear traverses of figures are not necessarily true for the linear traverses of the corresponding reciprocal figures.

For the reciprocation of such theorems we evidently require to find a process for the reciprocal figure which is not a linear traverse but the true reciprocal of a linear traverse.

The *linear traverse* of a finite portion of a straight line may be defined as the movement of a point from a fixed position on the line to another fixed position upon it, the moving point always remaining upon the straight line.

The reciprocal of this is the movement of a straight line from a fixed position through a given point to another fixed position through the same point, the moving line always passing through the given point. The moving line therefore revolves on a fixed point through a given angle from one position to the other; it may be therefore said to traverse the angle between its limiting positions, and thus the reciprocal of a linear traverse is an *angular traverse.*

Let $A\ B$ be a straight line upon which are two points A and B.



In the reciprocal figure let (A) be the straight line corresponding to A, and (B) that corresponding to B. Then $(A\,B)$, the intersection of these lines, corresponds to the line $A\,B$.

To every point on the line $A\,B$ between A and B, that is to every point on $A\overline{\ }B$, there corresponds a line through $(A\,B)$ comprised between one of the angles between (A) and (B); let this angle be $(A)\widehat{\ }(B)$. Thus to the successive positions which a point takes in moving from A to B along the line $A\overline{\ }B$ correspond the successive positions which a line takes in moving from the position (A) to the position (B), while it always passes through the point $(A\,B)$. Thus to the traverse of $A\overline{\ }B$ corresponds the traverse of the angle $(A)\widehat{\ }(B)$, and $A\overline{\ }B$ corresponds to $(A)\widehat{\ }(B)$, that is

$$A\overline{\ }B \asymp (A)\widehat{\ }(B).$$

Hence a more accurate expression of correspondence between reciprocal figures than the usual one seems to be that the position of the point $(A\,B)$ corresponds to the position of the line $A\,B$, while a given angle at $(A\,B)$, viz. $(A)\widehat{\ }(B)$, corresponds to a given finite segment $A\overline{\ }B$ of the line $A\,B$.

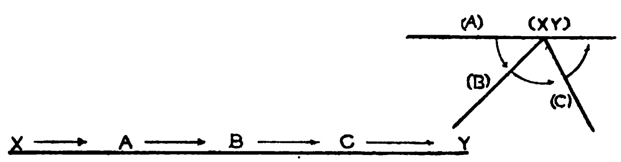
This, together with the other correspondences in the figure, may be expressed thus:

$$A\,B \asymp (A\,B)$$
$$A\overline{\ }B \asymp (A)\widehat{\ }(B)$$
$$A \asymp (A)$$
$$B \asymp (B).$$

Hence if two finite straight lines $A B$ and $C D$ are in the same straight line with one another, while one point $(A B C D)$ corresponds to the position of the one straight line on which they are, there will be two angles at the point corresponding to the two finite segments $A \overline{\ } B$ and $C \overline{\ } D$, viz. $(A)\widehat{\ }(B)$ and $(C)\widehat{\ }(D)$.

§ 51. *Traverse of an 'infinite segment' of a straight line. 'Point at infinity.'*

Let $X Y$ be a straight line upon which are two fixed points A and B.



Let $(X Y)$ be the point corresponding to $X Y$, and to the segment $A \overline{\ } B$ let the angle $(A)\widehat{\ }(B)$ correspond at $(X Y)$.

Let a linear traverse along $X Y$ proceed from A to B and from B to any point C, then the corresponding angular traverse through $(X Y)$ passes from (A) to (B) and from (B) to a line (C) passing through $(X Y)$ such that $B \overline{\ } C \backsimeq (B)\widehat{\ }(C)$. The line (C) evidently lies in the supplement of $(A)\widehat{\ }(B)$, and if the angular traverse is continued in the same direction from (C), the linear traverse will begin to travel from C in the same direction as before, that is, towards Y. Let the angular traverse continue till the remainder of the supplement of $(A)\widehat{\ }(B)$ is exhausted. As the angular traverse ends by the coincidence of the traversing line with (A), so the linear

traverse must end by the coincidence of the traversing point with A. And the traversing point must evidently approach A from the direction of X.

Hence the traversing point passing from B 'to infinity' on the one side of AB returns 'from infinity' to A on the other side of AB. We are thus led to the ordinary conception of the coincidence of the 'point at infinity' on a line in one direction from a fixed point upon it, with the 'point at infinity' upon it in the opposite direction from the same point.

Accordingly the traverse of the supplement of $(A)^\frown(B)$ may be said to correspond to an infinite traverse upon XY from B 'to infinity' on one side of AB, returning to A 'from infinity' upon the other side of AB; that is, to what may be called the 'infinite segment' between A and B, which comprises all the line except $A^{-}B$. If this 'infinite segment' is denoted by A_B the supplement of $(A)^\frown(B)$ may be correspondingly denoted by $(A)_\smile(B)$, and the above result expressed by the equations of correspondence

$$(A)^\frown(B) \backsimeq A^{-}B$$
$$(A)_\smile(B) \backsimeq A\_B$$

or if we put the points in the order of the traverse

$$(B)_\smile(A) \backsimeq B\_A.$$

Thus the application of the principle of duality to traverses not only leads to a new kind of traverse, the angular traverse of a figure, but also to the familiar conception that there is only one 'point at infinity' upon a straight line. The latter conception is not in any way suggested by linear traverses considered by themselves.

To the infinite traverse of what may be called 'the

whole' line, that is to the traverse of $A\overline{\ }B$ together with A_B, or if we put the points in the order of the direction of the traverse, $A\overline{\ }B$ together with B_A, which may be written $A\overline{\ }B\_A$, will then correspond an angle of 180° at the point (AB).

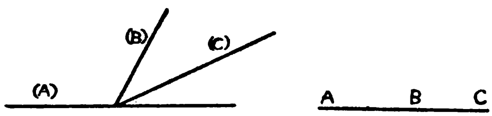
Thus there is a sense in which the point (AB) does not correspond to the 'whole' line AB, the true correspondent being the angle of 180° at (AB); and, as before, we should rather say that the position of the point (AB) corresponds to the position of the line AB.

§ 52. *Equivalence of finite and infinite linear traverses, and of their angular reciprocals.*

In a linear traverse, if the traverse of B_A was allowed as well as the traverse of $A\overline{\ }B$, it is clear that any figure could be exhausted in one traverse, and that for the purpose of traversing, the effect of adding the traverse of B_A to that of $A\overline{\ }B$ is exactly the same as traversing $A\overline{\ }B$ twice. As the double traverse of $A\overline{\ }B$ is excluded, the traverses of $A\overline{\ }B$ and A_B cannot both be admitted: they must be considered as equivalent, and as alternatives. Thus the traverse of $A\overline{\ }B$ excludes the traverse of B_A (or A_B), and conversely. Similarly the traverse of the angle $(A)^\frown(B)$ must be considered as excluding the traverse of $(A)_\smile(B)$.

But though the traverse of $A\overline{\ }B$ is in this sense alternative to that of A_B, it is not always indifferent in any particular figure which of them is chosen, as will be shewn hereafter, and similarly for the traverse of the angles $(A)^\frown B$ and $(A)_\smile(B)$.

It must be observed that the traverse of an angle $(A)^\wedge(B)$ does not exclude the traverse of an angle

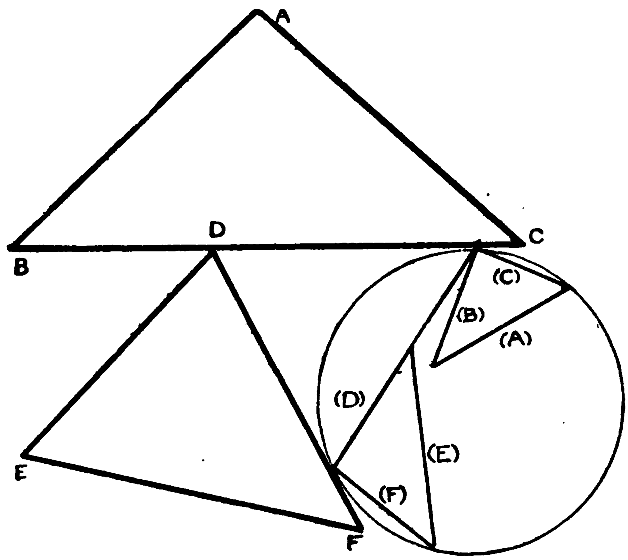


$(B)^\wedge(C)$ or $(B)_\vee(C)$ which is a *part* of its supplement $(A)_\vee(B)$, just as the traverse of the finite straight line $A\overline{\ }B$ does not exclude the traverse of the finite line $B\overline{\ }C$ which is on the same straight line with itself. Of course the sum of such angles as $(B)^\wedge(C)$ must be less than $(A)_\vee(B)$.

§ 53. *Nature of the correspondence between the lines of a figure and the angles of its reciprocal.*

It is convenient now to call the figure to which the linear traverse applies the *linear figure*, and its reciprocal, to which the angular traverse applies, the *angular figure.*

It will be obvious that the lines of the linear figure do not necessarily correspond to what may be the interior angles of the angular figure, but a given line may have for its correspondent the supplement of such an angle. Conversely an interior angle of the angular figure may correspond to the infinite segment joining the two points which correspond to the two sides of the angle: which angle corresponds to which side is determined by the mode of construction of the reciprocal figure.



Thus in this figure, where poles and polars are taken with reference to the circle, if the linear figure is taken to be (DE) (EF) (DF) (DE) (BCD) (AC) (BA), and the angular $ABCDEF$, the correspondences are as follows:

$$CB\,\widehat{\ }\,BA \backsimeq (CB)\overline{\ }(BA) \qquad CB\,\underset{\vee}{\ }\,BA \backsimeq (CB)\underline{\ }(BA)$$

$$BA\,\widehat{\ }\,AC \backsimeq (BA)\underline{\ }(AC) \qquad BA\,\underset{\vee}{\ }\,AC \backsimeq (BA)\overline{\ }(AC)$$

$$AC\,\widehat{\ }\,CB \backsimeq (AC)\overline{\ }(CB) \qquad AC\,\underset{\vee}{\ }\,CB \backsimeq (AC)\underline{\ }(CB)$$

$$FE\,\widehat{\ }\,ED \backsimeq (FE)\underline{\ }(FD) \qquad FE\,\underset{\vee}{\ }\,ED \backsimeq (FE)\overline{\ }(ED)$$

$$ED\,\widehat{\ }\,DF \backsimeq (ED)\overline{\ }(DF)$$

$$DF\,\widehat{\ }\,FE \backsimeq (DF)\overline{\ }(FE)$$

$$ED\,\widehat{\ }\,DB \backsimeq (ED)\overline{\ }(DB)$$

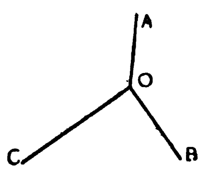
$$CD\,\widehat{\ }\,DF \backsimeq (CD)\underline{\ }(DF) \qquad CD\,\underset{\vee}{\ }\,DF \backsimeq (CD)\overline{\ }(DF)$$

§ 54. *Some definitions.*

For the sake of clearness definitions which relate to angular traverses will be put side by side with the corresponding ones for linear traverses.

Linear figure.

Two lines of the figure are *adjacent* if one extremity of the one coincides with one extremity of the other.



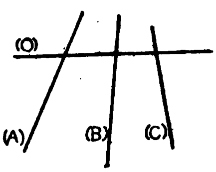
Thus the lines $O^{-}A$, $O^{-}B$, and $O^{-}C$ are adjacent, for an extremity of each of them coincides with the point O.

Two adjacent lines of the figure may be traversed continuously, for the traverse of the line $O^{-}A$, for instance, may begin from the point A and end at the point O, while the traverse of the line $O^{-}B$ may begin from the point O and end at the point B.

A *linear path* consists of several lines of the figure which are successively adjacent

Angular figure.

Two angles of the figure are *adjacent* if one of the containing lines of the one coincides with one of the containing lines of the other.



Thus the angles $(O)^\wedge(A)$, $(O)^\wedge(B)$, and $(O)^\wedge(C)$ are adjacent, for a containing line of each of them coincides with the line (O).

Two adjacent angles of the figure may be traversed continuously, for the traverse of the angle $(O)^\wedge(A)$, for instance, may begin from the line (A) and end at the line (O), while the traverse of the angle $(O)^\wedge(B)$ may begin from the line (O) and end at the line (B).

An *angular path* consists of several angles of the figure successively adjacent to one

to one another; that is, each of them has one extremity coinciding with an extremity of the one next before it in order, and the other coinciding with an extremity of the one next after it.

No two lines of a linear path have a common segment.

another; that is, each of them has one of its containing lines coinciding with a containing line of the angle next before it, and the other coinciding with a containing line of the angle next after it.

No two angles of an angular path have a common part.

It is obvious from the preceding that both a linear path and an angular path can be traversed continuously.

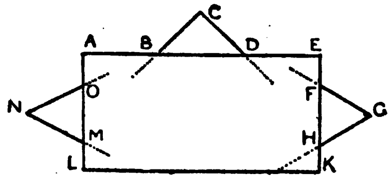
A *linear circuit* is a path such that the traversing point in a complete traverse of it returns to the position from which it started.

Thus if any line in the circuit is taken as the first line traversed, the last line traversed has its last point or extremity coincident with the first extremity of the first line.

An *angular circuit* is a path such that the traversing line in a complete traverse of it returns to the position from which it started.

Thus if any angle in the circuit is taken as the first angle traversed, the last angle traversed has its last containing line coincident with the first containing line of the first angle.

From the definition of an angular circuit, it is evident that an angular circuit need not yield a figure which is a linear circuit. For example, the following figure with angular points $A, B, C, D, E, F, G, H, K, L, M, N, O,$ is an angular circuit, but the figure formed by the lines not dotted is not a linear circuit.

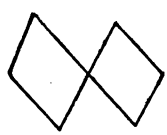


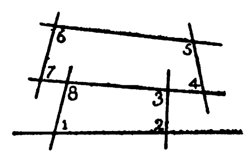
A *linear complex circuit* is such that the traversing point in a complete traverse of it returns to one or more points, which are points of the figure, beside the one from which it started.

An *angular complex circuit* is such that the traversing line in a complete traverse of it returns to one or more positions, which are lines of the figure, beside the one from which it started.

Thus a complex linear circuit must have one or more points, each of which is the common extremity of more than two lines of the figure.

Thus a complex angular circuit must have one or more lines, each of which is the common containing line of more than two angles of the figure.





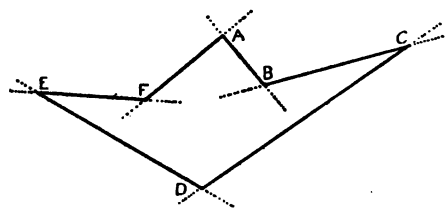
Here the line joining the angular points 4 and 7 is common to the four angles 7, 8, 3, 4. *See also* § 59.

A *linear single circuit* is such that the traversing point, in a complete traverse of it, returns to that point only from which it started, and each point of the figure is the common extremity of two lines only.

An *angular single circuit* is such that the traversing line, in a complete traverse of it, returns to that position only from which it started, and each line of the figure is the common containing line of two angles only.

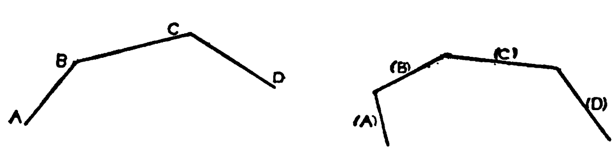
It must be noticed that as in a linear figure we do not necessarily join its angular points in every possible way, so we do not in an angular figure necessarily include in the angular points of it every intersection of its lines. Thus the following figure, of which the angular points are only *A*, *B*, *C*, *D*, *E* and *F*, no other intersections of the

lines being counted as angular points, is an angular single circuit.



By the *linear traverse* of a figure, or of a part of a figure, is understood the continuous traverse (as in § 1) of a linear path, no line of the path being traversed twice. Compare § 52.	By the *angular traverse* of a figure, or of a part of a figure, is to be understood the continuous traverse of an angular path, no angle of the path being traversed twice.
A traversed line is said to be *exhausted*.	A traversed angle is said to be *exhausted*.
A point is exhausted when every line of the figure radiating from it is exhausted.	A line is exhausted when every angle starting from it is exhausted; that is, when every angle of the figure at every angular point on the line is exhausted.
A figure is exhausted when every point of it is exhausted.	A figure is exhausted when every line of it is exhausted.

Let the points A, B, C, D, be joined by the straight lines AB, BC, CD. These form a linear path, and the corresponding lines (A), (B), (C), (D) enclose angles



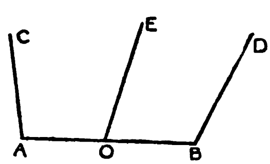
which form an angular path. The linear path consisting of the lines $A{}^{-}B$, $B{}^{-}C$, $C{}^{-}D$, may be written $A{}^{-}B\;B{}^{-}C$

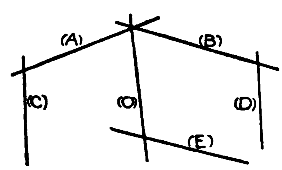
$C^{-}D$ or more conveniently $A^{-}B^{-}C^{-}D$; similarly the angular path may be written $(A)^{\frown}(B)^{\frown}(C)^{\frown}(D)$.

§ 55. '*Denomination*' *of the lines of an* '*angular figure.*' *Ambiguous case.*

If from a point in a linear figure radiate n lines, from the corresponding line of the angular figure there will start n angles; but before we can apply the conception of 'denomination' to the lines of the angular figure, it will be necessary to make some restrictions in the kind of linear figure of which the angular is the reciprocal.

Suppose that in a linear figure two at least of the lines which radiate from a given point of the figure are in the same straight line, then there will be at least three points of the figure in the same straight line; and, as will be shewn, there will be an ambiguity as to the number of angles which may be said to start from the line in the angular figure which corresponds to the given point.

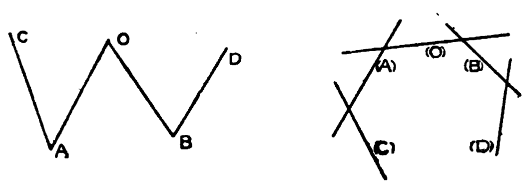




Let A, O, and B be three points of the linear figure in the same straight line. Suppose, for simplicity, there is one other line OE at O beside OA and OB, one other line AC at A beside AO, and one other line BD at B beside BO. Though $A^{-}O$ and $O^{-}B$ have in the reciprocal figure only one point, (AOB), corresponding to their position on the same straight line AOB, there is a separate angle corresponding to each, $(A)^{\frown}(O)$ to $A^{-}O$ and $(O)^{\frown}(B)$ to $O^{-}B$, and so far there is no

ambiguity. But these two angles have a common vertex $(A\,O\,B)$. Consequently the line (A) may be considered to have not only the two angles $(A)\widehat{\ }(C)$ and $(A)\widehat{\ }(O)$ starting from it, corresponding to the two lines $A\bar{\ }C$ and $A\bar{\ }O$ radiating from the corresponding point A, but also as having the angle $(A)\widehat{\ }(B)$ starting from it. And thus if a line is of even or of odd denomination, according as an even or odd number of angles start from it, (A) would not seem to be of the same denomination as A, and so the two figures not to correspond in denomination. Even if we could decide from the linear figure that the line (B) must be considered as a containing line of an angle starting from (O) and not from (A)—and we shall see hereafter that is not the true rule—we could not know this from the angular figure alone.

Similarly if two lines of the linear figure $A\bar{\ }B$ and $D\bar{\ }E$, which are not adjacent, are in the same straight line, the angles $(A)\widehat{\ }(B)$ and $(D)\widehat{\ }(E)$ have a common vertex $(A\,B\,D\,E)$, and there will be the same kind of ambiguity as to denomination.



If, however, $A\bar{\ }O$ and $O\bar{\ }B$ are not in the same straight line there is no ambiguity. The angles $(A)\widehat{\ }(O)$ and $(O)\widehat{\ }(B)$ have two different vertices $(A\,O)$ and $(O\,B)$, and the line (A) has two angles only, $(A)\widehat{\ }(O)$ and $(A)\widehat{\ }(C)$, starting from it corresponding to the two lines $A\bar{\ }O$ and $A\bar{\ }C$, which start from A.

A point in an angular figure which is the vertex of more than one angle of the figure, may be called an *ambiguous point.*

If in the linear figure no two lines, whether adjacent or not, are in the same straight line, it is evident that in the angular figure there will be on any line (X) exactly the same number of different angular points of the figure, at each of which one angle, and one only, starts from (X), as there are lines starting from the point X in the linear figure.

The investigation will, in the first instance, be confined to figures of this kind. Thus the linear figures to be now considered will have no two straight lines, whether adjacent or not, in the same straight line, and the corresponding angular figures will have no two angles with a common vertex.

For these figures 'denomination' may be defined as follows:—

Linear figure.	Angular figure.
An *odd point* is one at which there are an odd number of lines which radiate from the given point.	An *odd line* is one on which there are an odd number of angular points of the figure, at each of which one angle starts from the given line.
An *even point* is one from which an even number of lines radiate.	An *even line* is one on which there are an even number of such angular points.
An 'even point' is said to be of even *denomination*, and an 'odd point' of odd denomination.	An 'even line' is said to be of even *denomination*, and an 'odd line' of odd denomination.

For the angular traverses of angular figures restricted to the kind described, a system of theorems may be developed exactly corresponding to that which has been established for the linear traverses of linear figures.

§ 56. *Correspondence of linear and angular paths.*

If any part of a linear figure with finite lines can be exhausted in one traverse, this part forms a linear path. The corresponding part of the angular figure, being evidently an angular path, can be exhausted in one angular traverse. An angular path in an angular figure must have a linear path corresponding to it in the reciprocal linear figure, but this latter may contain infinite segments, as is evident from § 51. If the figures are unambiguous, the supplement of an angle may always be traversed instead of the angle itself, and thus an angular path between two lines can always be such as to have corresponding to it in the reciprocal figure a linear path consisting of finite segments.

In the case, however, of an ambiguous figure, as will appear later (§ 67), the angular traverses, with a certain beginning and end, which exhaust the figure, may all be such that the corresponding linear paths in the reciprocal figure all contain infinite segments.

§ 57. *Infinity of lines belonging to the angular figure.*

Let C_1 and C_2 be two linear figures. Let (C_1) and (C_2) be the corresponding angular figures.

Suppose C_1 and C_2 have a point X in common, then the reciprocal of this is a line (X) which is not necessarily a line of the figures (C_1) and (C_2).

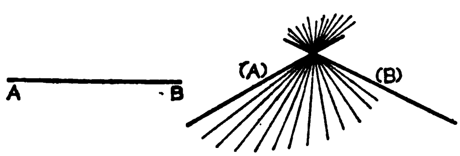
For example in Fig. 7 of § 49: the reciprocals of the triangles ABC and DEF are the triangles $(A)(B)(C)$ and $(D)(E)(F)$. The first two triangles intersect in the points X and Y, and the reciprocals of these points, the lines

(X) and (Y), are not lines of either of the figures $(A)(B)(C)$ or $(D)(E)(F)$.

Thus it seems as if, in a sense, the principle of duality did not hold here, and it were not true that if two circuits have a point in common their reciprocals must have a line in common.

But our ordinary way of viewing reciprocal figures requires to be supplemented by something which properly belongs to it and then the apparent exception disappears.

Suppose the points A and B are joined by the finite



straight line AB. We have seen already that to the line AB comprised between A and B corresponds the angle $(A)\,\widehat{\ }\,(B)$. Now all the points on $A\,\overline{\ }\,B$ are accounted points on the linear figure. To each of these corresponds a line through the point (AB) within the angle $(A)\,\widehat{\ }\,(B)$ which corresponds to $A\,\overline{\ }\,B$. Hence as each of the points which can be taken upon $A\,\overline{\ }\,B$ is accounted a point on the linear figure AB, so every line which can be drawn through the point (AB) within the angle $(A)\,\widehat{\ }\,(B)$ should be accounted a line on the angular figure.

Thus in general the lines on a reciprocal figure are not merely what we have called 'the lines of the figure,' viz. those which contain the angles of the figure, but also every line which can be drawn through the angular points within those angles at them which correspond to the lines of the linear figure.

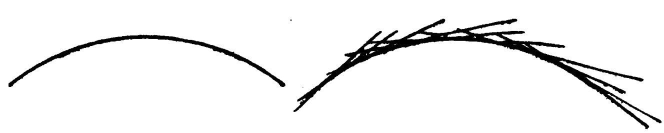
Now if the line $H\,\overline{\ }\,K$ of the figure C_1 has a point X in common with the line $L\,\overline{\ }\,M$ of the figure C_2, the corre-

sponding line (X) must pass through (HK) within the angle $(H)\widehat{\;}(K)$ which corresponds to $H\bar{\;}K$, and therefore (X) is a line on the figure (C_1). Also (X) must pass through (LM) within the angle $(L)\widehat{\;}(M)$ corresponding to $L\bar{\;}M$. Thus (X) is a line on the figure (C_2).

Hence if two linear figures have a point in common their angular reciprocals have a line in common.

In the traversing of curvilinear figures, the linear path traversed on a curve between the extremities of an arc has corresponding to it in the reciprocal figure the angular space swept over by the tangent of the reciprocal curve, as it moves from the position corresponding to one extremity of the given arc to the position corresponding to the other extremity.

Every tangent between the two given positions must be considered as a line upon that part of the reciprocal figure which corresponds to the given arc.



Thus the reciprocal of a linear figure must be considered as having an infinity of straight lines upon it; if rectilinear, the infinity drawn through each angular point, and if curvilinear the infinity of the tangents.

It will be convenient to call every point on a line of a linear figure, whether angular point or not, a point *on* the figure; what have been called points *of* the figure being for clearness designated as angular points of the figure (*see* § 48). So also any line passing through an angular point of an angular figure within the angle which belongs to the figure considered as traversed by an angular traverse, may be called a line *on* the figure.

§ 58. *Omission of angular paths, and denomination of remaining lines of the figure.*

The omission of angular paths may be illustrated by an example. Fig. 1 is linear, Fig. 2 type of its reciprocal.

Fig. 1.

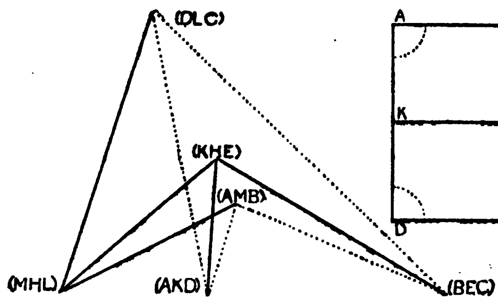
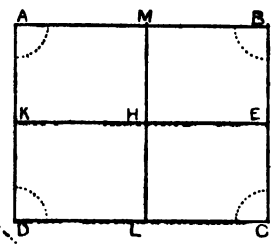


Fig. 2.



Linear figure.

Omit the linear path $(AMB)\bar{}(AKD)\bar{}(DLC)\bar{}(CEB)\bar{}(AMB)$.

This necessitates the retention of the point (AMB) as extremity of the line $(MHL)\bar{}(AMB)$,

of (AKD) as extremity of the line $(AKD)\bar{}(KHE)$,

of (DLC) as extremity of the line $(DLC)\bar{}(MHL)$,

of (CEB) as extremity of the line $(CEB)\bar{}(KHE)$,

of (KHE) and (MHL) as extremities of the line $(KHE)\bar{}(MHL)$.

Thus all the 'points of the figure' are retained.

The four lines lost are shewn by the dotted lines.

Angular figure.

Omit the angular path $AMB^\frown AKD^\frown DLC^\frown CEB^\frown AMB$.

This necessitates the retention of the line AMB as a containing line of the angle $MHL^\frown AMB$,

of AKD as a containing line of the angle $AKD^\frown KHE$,

of DLC as containing line of the angle $DLC^\frown MHL$,

of CEB as containing line of the angle $CEB^\frown KHE$,

of KHE and MHL as containing lines of the angle $KHE^\frown MHL$.

Thus all the 'lines of the figure' are retained.

The four angles lost are shewn by the dotted lines.

It is thus obvious that if we strike out a path in a linear figure, all the lines in the path are taken away, but those points which are the extremities of lines thus struck out, and are also extremities of lines not struck out, will remain.

Similarly if in an angular figure we strike out an angular path, every containing line of an omitted angle which is also a containing line of an angle not omitted, must remain, and thus it may happen that no line whatever can be omitted. Thus in the illustration above considered, no line can be left out of the angular figure. On the other hand if in the corresponding linear figure the corresponding linear path is struck out, lines will be left out. It will be found, of course, that though no lines are omitted in the angular figure, the retention of the lines corresponds exactly to the retention of corresponding points in the linear figure.

The denomination of a line is not altered in an angular figure, which is of the non-ambiguous kind described above (§ 55, page 85), if the angles exhausted by any angular traverse which passes through it are struck out. For the traverse exhausts one angle at an angular point on the line on arriving at the line, and another at another point on leaving it, hence the remaining number is even or odd according as the original number was even or odd.

This is the reciprocal of the theorem which has been proved for the traverses of linear figures in Part I, § 2, 4 (page 7); and it may be shewn also by a method reciprocal to that which is employed there, that if an angular path is struck out the line which is the first position of the traversing line and the line which is its last position are either exhausted or changed in denomination.

§ 59. *Traverse of angular complex circuits.*

If two linear single circuits C_1 and C_2 have a point X in common, their reciprocals, the angular figures (C_1) and (C_2), have a line (X) in common.

Suppose we begin a linear traverse of the complex circuit C_1 C_2 from a point A in C_1. When we arrive at the common point X we traverse the whole circuit C_2 from X back to X, and then continue the traverse of C_1 till we arrive again at A. In this way the complex C_1 C_2 is exhausted in one traverse.

Similarly if we begin the angular traverse of the complex (C_1) (C_2) from a line (A) in (C_1), when we arrive at the common line (X) we traverse the whole angular circuit (C_2), from (X) and back to (X), and then continue the angular traverse of (C_1) from (X) till we arrive again at (A). Thus (C_1) (C_2) is exhausted in one traverse, and is an angular complex circuit.

If the circuits C_1 and C_2 have another point Y in common, the lines through this belonging to C_1 are exhausted in the traverse of C_1, and the lines through it belonging to C_2 in the traverse of C_2.

Similarly if (C_1) and (C_2) have another line (Y) in common, the angles starting from this in the circuit (C_1) are exhausted in the traverse of the circuit (C_1) and those starting from it in (C_2) in the traverse of C_2.

If there is a system of mutually exclusive angular circuits, that is, circuits such that no two have a part of any angle at their angular points in common, then if each is connected with some other in the above manner, and they thus form a continuously connected figure, it may be shewn in the same way that the system can be exhausted in one traverse. Such a system is a complex circuit.

§ 60. *Traverses which exhaust odd lines, and traverses which exhaust even lines.*

The theorems that an angular figure cannot have one odd line only, and that odd lines, if they occur, must occur in pairs: that an even-line figure is an angular circuit, and therefore can be exhausted in one traverse; and that a figure with $2n$ odd lines requires n traverses to exhaust it, may be proved by arguments which are the reciprocals of the arguments by which the corresponding theorems for linear figures have been proved.

But some different forms of the proofs may be given, which in turn can be reciprocated for linear figures; and besides, it will be necessary to explain the exact form of the reciprocal argument in some places.

A traverse which begins from a line but does not end at it, or ends in a line from which it does not begin, whether in either case it also passes through them or not, clearly exhausts an odd number of the angles which start from the line, and if the path of the traverse is struck out the denomination of such lines is changed.

A traverse which begins and ends at the same line has evidently the same effect as a traverse which passes through it: it exhausts an even number of angles, and therefore if it is struck out the denomination of the line is not altered.

Hence it follows that an odd line cannot be exhausted by a traverse which passes through it, nor can a traverse which exhausts it and ends at it be one which begins at it. And an even line cannot be exhausted by a traverse which begins from it and does not end at it, nor can a traverse which exhausts it and ends at it not begin from it.

§ 61. *The number of odd lines in an odd-line figure is even.*

Suppose a figure has one odd line. Let an angular traverse begin from it. This cannot end at an even line, for this line would have to be exhausted, but, by the above (§ 60), it cannot be exhausted by a traverse which does not begin from it.

It cannot end at the given odd line, for then this line would be exhausted; but, by the above, an odd line cannot be exhausted by a traverse which begins from it.

There must therefore be another odd line at which the angular traverse ends.

Proceeding then as in § 3, 2 it may be shewn that in an odd-line figure, odd lines must occur in pairs, that is, the number of odd lines must be even.

§ 62. *An even-line figure is an angular circuit.*

A traverse which starts from an even line in an even-line figure must, by the preceding, end at the line from which it started. The angles of this traverse form an angular circuit. If this traverse is struck out the figure remains an even-line figure, and a second traverse in the remaining figure from an even line will similarly end in the line from which it starts and form an angular circuit. And thus, in a manner corresponding to the second proof in § 4, it may be shewn that the figure consists of a number of angular circuits.

In the nature of the case none of these has any angular point in common with any other, since each circuit is formed after all the previous circuits have been struck out. Whatever points of intersection, therefore, they may

have, these cannot be any of the points counted as angular points of the figure (§ 54).

The original figure, however, is continuous, and this means in an angular figure that it consists of a system of adjacent angles. Hence different parts of the figure can only be connected with one another by means of common lines; which therefore join angular points of the one part of the figure to angular points of the other.

As the whole figure is continuous the circuits into which it has been resolved must be continuous with one another. Hence each one must have a line in common with some other, that is, an angular point of the one must be joined to an angular point of some other by a line which falls within an angle of the figure at each of the two angular points which it connects, or coincides with a containing line of such angle.

If two parts of a figure intersect in an angular point of either of them, that is, an angular point of the figure, they must have at least one common line, and at least one angle, or a part of it, of the one adjacent to one, or a part of one, of the other. The existence of such a point then would cause the two parts of the figure to be continuous. On the other hand, if it is not an angular point of the figure, its existence does not make the given parts of the figure continuous.

The intersections then of the circuits now considered not being, as has been shewn, angular points of the figure, do not make them continuous with one another, but they are made continuous by lines joining their angular points in the manner above described.

It follows from § 59 that the traverse of any one of the circuits can be made continuous with that of all the rest. Hence the figure can be exhausted in one traverse and is an angular circuit.

§ 63. *Examples of even-line figures, or angular circuits.*

The following are examples of angular circuits:—

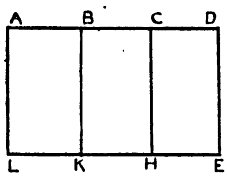


Fig. 1.

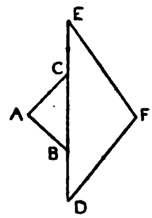


Fig. 2.

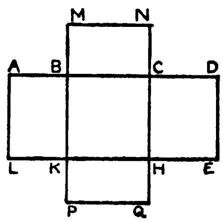


Fig. 3.

It will be observed that Fig. 1 taken as linear is not a circuit. It would have four odd points and require two traverses.

So also the angular circuit in Fig. 2 if taken as linear would have two odd points.

Fig. 3 if taken as angular is an angular circuit, and if taken as linear is a linear circuit.

If two figures, which taken by themselves are linear circuits, have a line or part of a line in common, the complex figure cannot be traversed as one circuit, for evidently the common line or part of line would have to be traversed twice if the traverse is to return to the point from which it started.

Similarly if two angular circuits have an angular point in common, which is an angular point in each, they cannot be traversed as one complex angular circuit, for if the traverse is to return to the line from which it started an angle at the common point must be traversed twice.

Thus the following figures 4 and 5, which are linear circuits, are not angular circuits, if A, B, C, D and E are angular points of them.

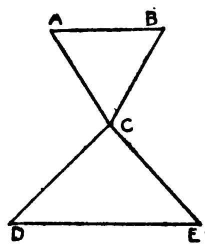


Fig. 4.

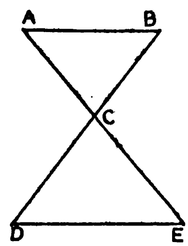


Fig. 5.

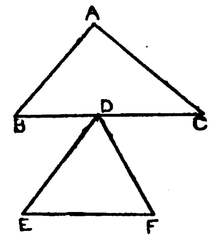


Fig. 6.

Fig. 4 is an ambiguous figure, and it will be shewn hereafter (§ 71) that it must be considered as a figure with two odd lines, and so cannot be traversed as a circuit.

Fig. 5 becomes an angular single circuit if the angle at C is not counted as an angle of the figure.

In Fig. 6 the point D is common to the two figures $A\,B\,C$ and $D\,E\,F$, but it is an angular point in one of them, $D\,E\,F$, only, and the figure is an angular circuit.

It consists, in fact, of two mutually exclusive angular single circuits $A\,B\,C$ and $D\,E\,F$, which have a common line $B\,D\,C$ to connect them, and therefore is a complex circuit.

$B\,D\,C$ is a common line because in the circuit $D\,E\,F$ the angle at D is not $E\,D\,F$ but its supplement, $B\,D\,C$ therefore is a line in the angular figure $D\,E\,F$.

It will be found that if $E\,D\,F$ is taken as an angle of the figure, that is, as an angle which is to be traversed, any traverse which returns to the line from which it started will necessitate that the supplement of $E\,D\,F$ is traversed as well as $E\,D\,F$, which is excluded by § 52.

This figure also is of the ambiguous kind, and, as will appear from the theory of these figures, may be considered as having upon $B\,D\,C$ four angular points, from each of which one angle starts.

Thus all its lines are even lines, and it can be traversed as an angular circuit.

§ 64. *Figures with one or more pairs of odd lines.*

If a figure has two odd lines it follows from § 60 that a traverse T starting from one odd line must end at the other. It may be shewn, by the reciprocal of the method of § 6, in the manner explained in § 62, that the remainder of the figure consists of a system of mutually exclusive angular circuits, each of which is either connected with the path of T by a line or lines joining angular points in it to angular points in the path of T, or is part of a complex circuit which is connected in this way with T. Thus the figure can be exhausted in one traverse.

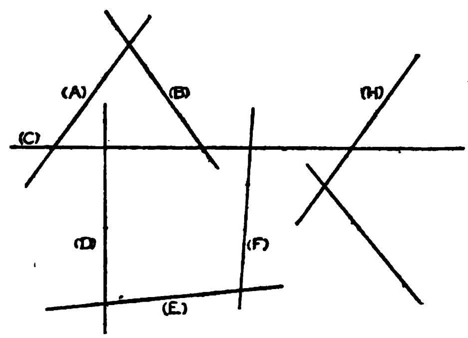
The conceptions of 'single paths' and 'accessories' apply to angular as to linear paths.

It may be shewn, as in § 27, that if an angular path with its peculiar accessories joining two odd lines, in a figure of more than two odd lines, is struck out, the remaining figure or figures must contain $2n-2$ odd lines, and the odd lines occur in each of the remaining figures in pairs. Hence it may be shewn, as in § 28, that the number of traverses required to exhaust an angular figure of $2n$ odd lines is n,

§ 65. *'Single angles.' 'Conjugate odd lines.'*

It may happen that some of the angles at the angular points of a given line may be traversed as parts of angular circuits.

If when these angles have been associated in the maximum number of mutually exclusive angular single circuits, angles remain over which do not belong to these circuits, such angles may on the analogy of 'single lines' be called 'single angles.'



In this figure, the line (C) has five angular points: $(A)^\frown(B)^\frown(C)^\frown(A)$ and $(C)^\frown(D)^\frown(E)^\frown(F)^\frown(C)$ are angular circuits: $(H)^\frown(C)$ is a single angle.

Fixed and variable 'single angles' may be defined in a manner analogous to fixed and variable 'single lines' and have properties corresponding to those proved for the latter in § 12.

It may be shewn, as in § 13, that every odd line has at least one single angle:—

As in § 16, that if there are *m* single angles at a line (O), the figure consists of *m* separate angular figures which have the line (O) in common but no other:—

As in § 17, that the maximum of single angles at a line (O) must be at least one less than the number of odd lines in the figure, if (O) is an odd line, and not greater than the number of odd lines if (O) is even:—

As in § 18, that an odd line which can be exhausted in a single angular traverse must have one single angle and no more, and conversely:—

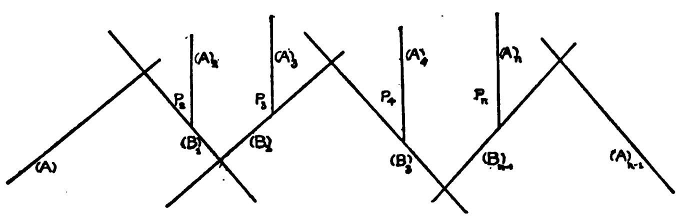
As in § 22, that if two odd lines are such that any angular path connecting them must have as a part of it a part of what must be an angular circuit at one or other of them

in any maximum resolution at that line, these two odd lines cannot be exhausted by the same traverse:—

As in § 25, that the maximum number of odd lines with more than one single angle, in a figure of $2n$ odd lines, is $n-1$, and the minimum of lines with but one single angle is $n+1$:—

As in § 26, that the minimum number of pairs of odd lines, each of which can be exhausted by the same traverse, that is of pairs of conjugates, is $\frac{n(n+1)}{2}$.

The figure, corresponding to that in § 26, which illustrates the case of the minimum number of pairs of conjugate odd lines is this:—



where P_1 is the angular path

$$(A_1)\frown(B_1)\frown(B_2)\frown\frown(B_{n-1})\frown(A_{n-1}),$$

P_2 the angular path $(A_2)\frown(B_1)$,
P_n the angular path $(A_n)\frown(B_{n-1})$.

§ 66. *Constructive method.*

The constructive method applies throughout to angular figures.

In this method, just as two linear paths are associated by the coincidence of a point upon one with a point in the other, so two angular paths are associated by the coincidence of a line belonging to the one with a line

belonging to the other. It must be remembered that such connecting line need not be a containing line of angles in the two angular paths, but may be any line which joins an angular point of the one path to an angular point of the other, and falls within the angle at each of these angular points which is proper to the path, and not within its supplement (§ 57).

Just as the linear paths are to have no common segment (§ 34), so the associated angular paths are to have no angle, or part of an angle, in common.

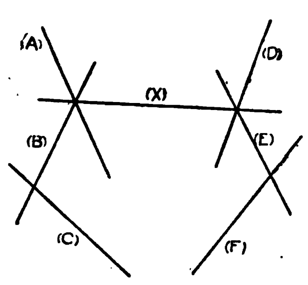


Fig. 1.

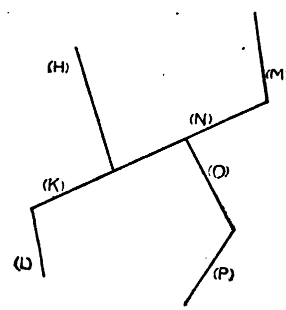


Fig. 2.

In Fig. 1 the angular paths

$$(A)\frown(B)\frown(C) \text{ and } (D)\frown(E)\frown(F)$$

are associated by the line (X) joining $(A B)$ and $(D E)$ and falling within the angles of the paths at these points. In Fig. 2 the angular paths

$$(H)\frown(K)\frown(L) \text{ and } (M)\frown(N)\frown(O)\frown(P)$$

by the coincidence of (K) and (N).

It may be proved, as in § 33, that if a figure can be exhausted in one traverse it must either have all its lines even, and be also a circuit, or that it has only two odd lines.

The remainder of the theorems in the constructive method for linear figures can be reciprocated for angular figures.

§ 67. *Relation of the traverses of an ambiguous angular figure to those of its linear reciprocal.*

The angular figures of which the linear reciprocals have two or more of their lines in the same straight line will now be considered.

In these angular figures there will be, as we have seen, angles which have a common vertex.

It will first be shewn that the denomination of the lines in such figures for the purpose of traversing is not necessarily determined from the denomination of the corresponding points of the linear figure, in which the lines are considered finite.

Suppose that in the figure of § 55 in order to make the denomination of (A) correspond to that of A, (A) is considered as having only the angles $(A)\widehat{\ \ }(O)$ and $(A)\widehat{\ \ }(C)$ starting from it

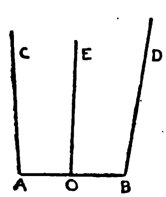


Fig. 1.

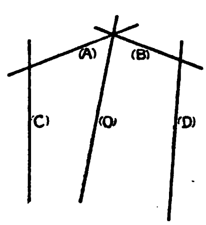


Fig. 2.

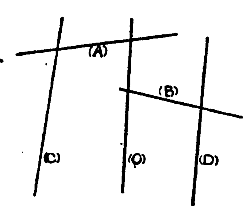


Fig. 3.

corresponding to the two lines $A\overline{\ \ }O$ and $A\overline{\ \ }C$ radiating from C, and that (B) is not treated as making an angle $(A)\widehat{\ \ }(B)$ with (A) but as making the angle $(O)\widehat{\ \ }(B)$ with (O), just as $A\,O$ and not $A\,B$ is considered as radiating from A.

It is obvious that the effect is the same as if the point in which (B) and (O) intersect were shifted from $(A\,O)$ as

shewn in Fig. 3. On this principle every line of the angular figure could be made of the same denomination as the corresponding point of the linear.

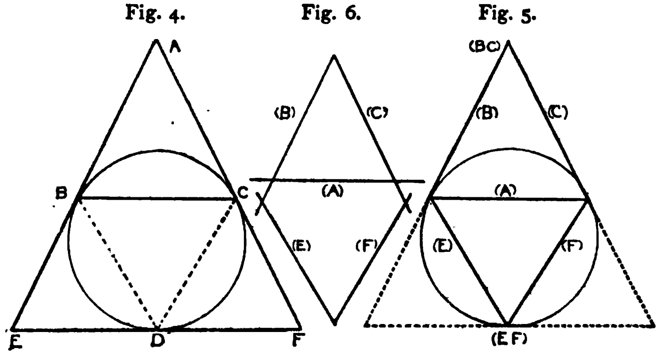
Suppose we have a linear figure of the kind now contemplated with two odd points only, this can be exhausted only by a traverse beginning with one of these points and ending in the other, that is, by a traverse between these points.

The corresponding angular figure would have two odd lines, and would be exhausted by a traverse between these two: and if this were the only way of determining denomination for the angular figure, just as the linear figure can only be exhausted by a traverse between its odd points, the angular figure could only be exhausted by a traverse between the two odd lines.

But there is nothing to necessitate that the relation of the angles to the lines of the angular figure could not be arranged so as to give a different angular traverse for the angular figure, to which, of course, there will correspond no finite linear traverse of the linear figure: nor indeed to shew that this figure may not be a circuit, and therefore have a traverse starting from each of its lines in turn.

It will be now shewn that this want of correspondence in the traverses is really possible.

Fig. 4. Fig. 6. Fig. 5.



Let AEF be a triangle with inscribed circle BDC. Join BC and take $ABCFE$ as the linear figure. Form the reciprocal figure as shewn in Fig. 5, by taking poles and polars with respect to the circle. The angular points of the reciprocal figure are (BC), (EF), (ABE), and (ACF).

The linear figure has two odd points, B and C, and, if we make the denomination of the lines in the angular figure correspond to that of the points in the linear, (B) and (C) will be odd lines, and the angular figure will be treated as if it had the modification shewn in Fig. 6. The latter figure has two odd lines (B) and (C), the rest being even, and it can only be exhausted by a traverse between the two lines (B) and (C).

But now Fig. 5 is an angular circuit, for it may have as reciprocal Fig. 7, as we have seen above, § 49, which

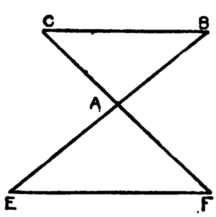


Fig. 7.

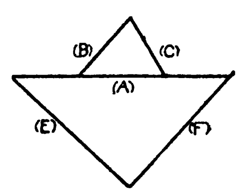


Fig. 8.

is a linear circuit with finite lines, and the reciprocal of a linear circuit must be an angular circuit (except in the case, of which this is not an example, described in § 68).

This may be verified in the figure itself by starting a traverse from any line in it, e. g.

$$(B)\frown(C)\frown(A)\frown(E)\frown(F)\frown(A)\frown(B)$$
$$(B)\frown(A)\frown(E)\frown(F)\frown(A)\frown(C)\frown(B)$$
$$(A)\frown(B)\frown(C)\frown(A)\frown(F)\frown(E)\frown(A)$$

&c. &c.

It can also be verified by considering the angles to start as in the modification shewn in Fig. 8. Every line of this latter figure is an even line. It is therefore an angular circuit, and hence the original figure is an angular circuit.

Thus an angular circuit, in the ambiguous case, may have a linear figure corresponding to it which is a linear circuit with finite lines or a figure which is not.

The figure of § 53 gives a similar case, if *DBACD FE* is taken for the angular figure.

Nevertheless when the ambiguous figure can be traversed as an angular circuit, inasmuch as no angle of it is traversed twice over there must correspond to it a traverse of the linear figure in which no line is traversed twice over. Hence in the case where the latter figure is not a circuit, considered as consisting of finite lines, it must be traversable as a circuit if for some of the finite segments infinite segments are substituted.

This may be illustrated by Figs. 4 and 5. (See above, p. 102.)

Fig. 4.

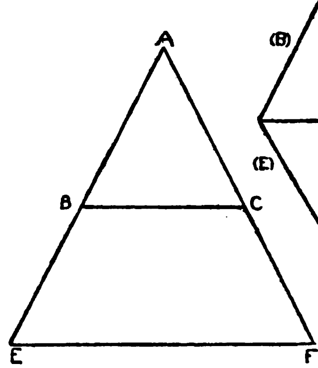


Fig. 5.

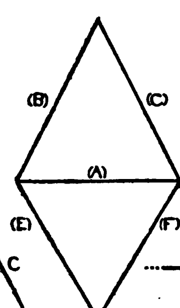


Fig. 9.

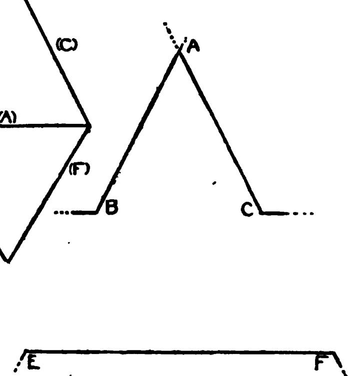


Fig. 5 may be traversed as an angular circuit, e.g. thus:—

$$(A)^\wedge(B)^\wedge(C)^\wedge(A)^\wedge(F)^\wedge(E)^\wedge(A)$$

The correspondences are (as the relation to the circle shews):—

$$\begin{gathered}
(A)^\wedge(B) \asymp A\,\overline{\ }\,B \\
(B)^\wedge(C) \asymp B\,\underline{\ }\,C \\
(C)^\wedge(A) \asymp C\,\overline{\ }\,A \\
(B)^\wedge(E) \asymp B\,\underline{\ }\,E \\
(E)^\wedge(F) \asymp E\,\underline{\ }\,F \\
(F)^\wedge(C) \asymp F\,\underline{\ }\,C \\
(A)^\wedge(F) \asymp A\,\underline{\ }\,F \\
(E)^\wedge(A) \asymp E\,\underline{\ }\,A \\[1ex]
(A)_\vee(B) \asymp A\,\underline{\ }\,B
\end{gathered}$$

&c., &c.

where the upper angle sign, as in $(A)^\wedge(B)$, is always taken to refer to an interior angle of the angular figure (Fig. 5), though, of course, the upper angle sign has not necessarily this meaning in itself.

Thus we have

$$(A)^\wedge(B)^\wedge(C)^\wedge(A)^\wedge(F)_\vee(E)^\wedge(A) \asymp A\,\overline{\ }\,B\,\underline{\ }\,C\,\overline{\ }\,A\,\underline{\ }\,F\,\overline{\ }\,E\,\underline{\ }\,A$$

Hence the linear figure is treated as having its points A and F united by an infinite segment. So also the points C and F are considered to be united by an infinite segment only. The same is true for the points A and E, the points B and E, and the points B and C (Fig. 9).

This has the effect of making every point of the linear figure an even point, and the figure itself is a circuit with some of its lines infinite.

It will be observed that this result may be obtained with BC as a finite segment; and accordingly it will be found that in the above traverse $(B)_\vee(C)$, of which the correspondent is $B\,\overline{\ }\,C$, may be substituted for $(B)^\wedge(C)$.

Another illustration may be given from the figure of § 53.

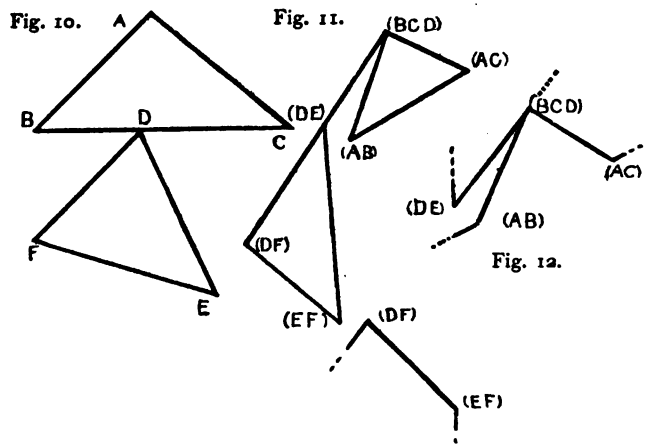


Fig. 10. Fig. 11. Fig. 12.

Fig. 11 is the linear figure, and Fig. 10 the angular. According to the correspondences which have been given in § 53,

$$FE \frown ED \frown DB \frown BA \frown AC \frown CD \frown DF \frown FE$$
$$\simeq (FE)\underline{\;}(ED)\overline{\;}(DB)\overline{\;}(BA)\underline{\;}(AC)\overline{\;}(CD)\underline{\;}(DF)\overline{\;}(FE).$$

Fig. 11 therefore is treated as shewn in Fig. 12, which has all its points even and is a linear circuit with infinite lines. With finite lines it would have two odd points.

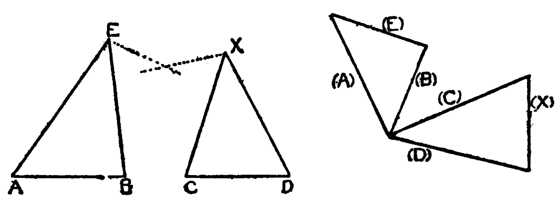
It will be observed that it is unnecessary for this purpose that (BA) (AC) and (FE) (ED) should be infinite segments, and accordingly it will be found that in the above traverse $BA \smile AC$ and $FE \smile ED$, which correspond respectively to $(BA)\overline{\;}(AC)$ and $(FE)\overline{\;}(ED)$, may be substituted for $BA \frown AC$ and $FE \frown ED$.

In such cases, then, every traverse of the ambiguous angular figure may be determined from the linear figure, if infinite segments are sometimes taken as alternative to the finite segments.

§ 68. *Case in which the traverses of an ambiguous angular figure do not correspond to those of its linear reciprocal.*

But there is a case in which an ambiguous angular figure cannot be said to have its traverses thus corresponding to those of its linear reciprocal.

Suppose two lines of the linear figure AB and CD, which are not adjacent, are brought into the same straight line without altering the denomination of any point in the figure (see § 69). In the angular figure the vertices of the angles $(A)\widehat{\ }(B)$ and $(C)\widehat{\ }(D)$ will coincide, and thus at what must be an angular point of the angular figure there will be introduced an angle, for instance $(B)\widehat{\ }(C)$,



to which corresponds no line of the linear figure, and hence one traverse at least of the angular figure will not have a corresponding traverse in the linear.

Further, as this angle must be considered as starting from two of the four lines (A) (B) (C) (D), or as part of an angle starting from two of these four, the denomination of two of the lines must be changed when (AB) comes to coincide with (CD). Thus, for instance, if the linear figure is a circuit, the denomination of each line in the angular figure before AB and CD came to be in the same straight line was even. Now, therefore, two are odd, and thus the figure is no longer an angular circuit, but a figure with two odd lines.

Thus, in the figure of § 53, let $DBACDFED$ be the linear figure, and therefore a linear circuit. Let EF come into the same straight line as AB (Fig. 1), the points (EF) and (AB) will then coincide (Fig. 2).

Fig. 1.

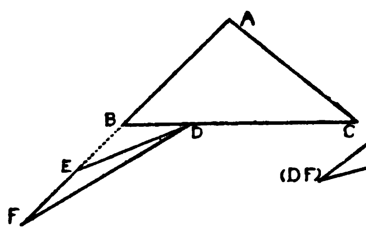
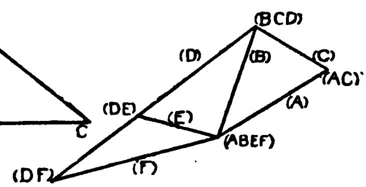


Fig. 2.



The angle $(B)\widehat{\ }(E)$ has no line to correspond in the linear circuit, and cannot be included in the traverse of the angular circuit $(A)(B)(C)(D)(E)(F)$.

The angular figure can only be traversed as a figure with two odd lines, which must be two out of the four lines (A), (B), (E), and (F).

If no angle in the angular figure, though at what must be an angular point in it, is to be counted which does not correspond to a line in the linear figure, the angle $(B)\widehat{\ }(E)$ would be excluded and the correspondence restored. But if we have not the linear figure before us we cannot make this restriction. Besides, given the angular figure, the reciprocal of it would include a line BE corresponding to $(B)\widehat{\ }(E)$. In any case it remains that from the reciprocation of Fig. 1 there necessarily arises a figure (Fig. 2), which, though only those of its angular points are counted as angular points of the figure which correspond to the lines of the linear figure, has not, taken in itself, the same system of traverses as Fig. 1.

§ 69. *In a linear figure lines may be brought into the same straight line without changing the denomination of any point.*

If two lines of a finite linear figure, of which no two are originally in the same straight line, are brought into the same straight line, and no two lines of the figure are thereby brought into coincidence, the nature of the traverse or traverses of the figure will not be altered: for obviously every point in it will have the same number of lines radiating from it as before, and its denomination, therefore, will be unaltered.

If the two lines are sides of a triangle of which the base is one line of the figure, and the vertex an angular point of the figure, this base will be brought into coincidence with the sides, and the denomination of the points at the extremities of the base will be altered. On the other hand, if the two lines, though adjacent, are not sides of such a triangle, the change in their direction need not involve a change in the denomination of any point in the figure.

Suppose that two lines of the figure which are not adjacent are brought into the same straight line: if one pair at least of their extremities is not joined by one line of the figure, the change may be made in such a way that the denomination of no angular point of the figure is altered.

If three straight lines which pass through the same point, this point being an angular point of the figure, are brought into the same straight line, one, at least, of them must be brought into coincidence with one of the others, or have a segment in common with it, and thus there will necessarily be an alteration of denomination.

§ 70. *Method of determining the traverses of an ambiguous angular figure.*

A linear figure, of which no two lines originally are in the same straight line, has an unambiguous angular reciprocal. If some of its lines are brought into the same straight line without altering the denomination of any point, and therefore without altering the nature of its traverses, the reciprocal is an ambiguous angular figure, and if no new angle (§ 68) has been introduced the traverses of the ambiguous figure can be determined from those of the unambiguous angular figure.

We may now give a general method for determining the traverses of an ambiguous figure considered in itself.

This may be found in two ways: first by the help of the theory of reciprocation, but without reference to a particular linear figure from which the reciprocal may have been constructed; and secondly, without any reference to reciprocation.

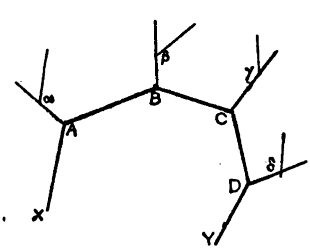


Fig. 1.

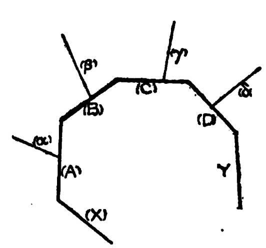


Fig. 2.

Suppose *AB*, *BC*, *CD*, &c., are adjacent lines of a linear figure which has no two of its lines in the same straight line: and suppose the polygonal contour *ABCD* is such that no two of its angular points which are not

adjacent are joined by a straight line which is one line of the figure.

Then the lines of the corresponding angular figure form a polygon, and none of the points of intersection (AB), (BC), (CD) coincide with one another.

Now if AB, BC, CD, &c., are brought into the same straight line with one another as in Fig. 3,

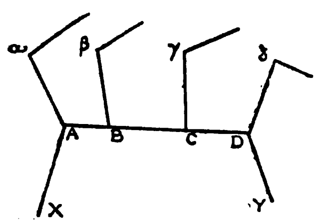


Fig. 3.

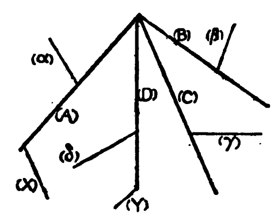


Fig. 4.

the points A, B, C and D remain of the same denomination as before, and the denomination of no other point of the figure need be changed.

Thus a modified figure can be made with all its points of the same denomination as those of the corresponding points in the original figure, and will have the same system of traverses.

In the angular reciprocal of the modified figure (Fig. 4), the lines (A), (B), (C), (D) pass through the same point. No angle is introduced which has no corresponding line in the linear figure, and the traverses of the modified angular figure correspond exactly to those of the modified linear figure. These latter are of the same system as those of the unmodified linear figure which again correspond to those of the unmodified angular figure. Hence the traverses of the modified angular figure (Fig. 4), which has the ambiguous form, are of exactly the same system as those

of the unmodified angular figure (Fig. 2) and can be determined from it.

It must be noticed that in forming the polygon, no line of it must intersect more than two of the others, for otherwise when the lines are brought into the same straight line, some of them would be two sides of a triangle and consequently the denomination of the points in the original figure (Fig. 1) will be altered.

Conversely if we have the lines (A), (B), (C), and (D) given as passing through the same point, as in Fig. 4, we may replace them by corresponding lines forming a polygon, as in Fig. 2, each of them meeting two of the others only, except the outside lines which meet one only of the others. Then the original angular figure which is ambiguous (Fig. 4) will be traversable exactly in the same way as the modification of it which is unambiguous (Fig. 2).

By varying the order of the lines in the sides of the polygon we shall produce every type of unambiguous angular figure, of which the given ambiguous angular figure is a modification, and thus determine all the possible traverses of the ambiguous figure.

We may arrive at the same result without the use of reciprocation. Displace the meeting-points of the lines (A), (B), (C), and (D) with one another from coincidence in $(ABCD)$ so that the lines form a polygon as in Fig. 2. Then it is clear that any order in which a line can pass by angular traverses from one line to another in the polygon has an order corresponding to it in the angular traverses of the original figure—the ambiguous case of Fig. 4.

Conversely the polygon can be made so as to correspond to any given traverse of the ambiguous figure.

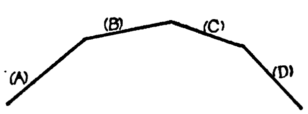
Hence by varying the order of the lines of the polygon we can produce every type of unambiguous figure which

corresponds to the various possible traverses of the ambiguous figure.

The same will hold if there is more than one angular point at which there are several angles, and we shall have to combine all the various orders of the sides of the polygon which replace one angle with those of the sides of all other such polygons.

Thus we can determine all the possible traverses of an ambiguous figure, and the different ways in which the denomination of its lines is to be estimated.

We shall determine the minimum number of traverses by which the ambiguous angular figure can be exhausted, and therefore, also, whether it can be an angular circuit or not if we determine the minimum number of odd lines which it can be considered to have: that is, if we find the corresponding unambiguous angular figure which has a minimum of odd lines.



Now in the polygon (A) (B) (C) (D), the first and last lines, A and D, have on them one angular point of the polygon, and the intermediate ones have two. But the order of the lines is arbitrary. Thus if the line (A) in the original ambiguous figure has an odd number of angular points beside that at the common vertex $(A\,B\,C\,D)$, we can make it an even line by constructing the polygon so that (A) is the first or last line of the polygon. And if a line has an even or odd number of angular points beside the common vertex, it will be an even or odd line

if it is an intermediate line of the polygon, or an odd or even if it is an outside line.

It is clear, then, that in order to determine the minimum of odd lines we must arrange the lines in the various polygons which replace the angular points so as to effect a maximum of even lines. If at any ambiguous point the lines are necessarily odd if we do not count the point itself, only two of them can be made even by the process described, and the figure cannot be an angular circuit. If they are all even, without the ambiguous point, two of them must be odd in the unambiguous figure. And they can only all be even in the unambiguous figure, if two only of them are odd in the ambiguous figure, the ambiguous point not being counted.

§ 71. *Examples of traverses of ambiguous figures. Proof that the linear reciprocal of an ambiguous angular figure is also ambiguous.*

We may take some examples—

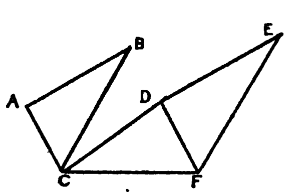


Fig. 1.

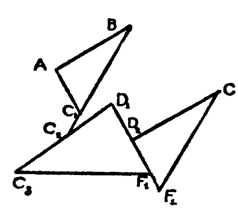


Fig. 2.

Let the angular figure be $ABCDFED$ with three ambiguous points C, D, and F.

If we arrange the polygons, or polygonal contours, at these points as shewn in Fig. 2, we get a figure with two odd lines BC_1C_2, $D_1C_2C_3$; and it may be verified that

Fig. 1 can be exhausted by a traverse which begins with BC and ends with CD, as if it were a figure with these two lines as odd lines. Thus to

$$C_1B^\frown BA^\frown AC_1^\frown C_1B^\frown D_1C_3^\frown C_3F_2^\frown$$
$$F_1D_2^\frown D_2E^\frown EF_1^\frown F_1D_1^\frown D_1C_3$$

in the unambiguous figure (Fig. 2) corresponds

$$CB^\frown BA^\frown AC^\frown CB^\frown DC^\frown CF^\frown$$
$$FD^\frown DE^\frown EF^\frown FD^\frown DC$$

in the ambiguous figure.

But at C, since CA and CB have one angular point each beside the ambiguous point, these two lines can be made even by becoming the outside lines of the polygon corresponding to C (Fig. 3).

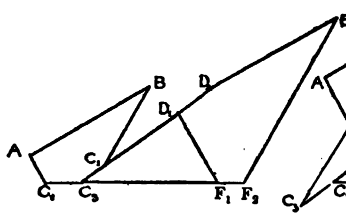


Fig. 3.

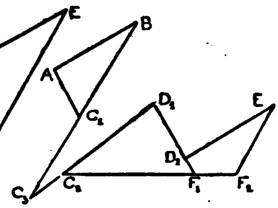


Fig. 4.

Then the line C_1D_1 corresponding to CD has two angular points C_1 and C_3, it will therefore be even if it is made inside line of the polygon corresponding to D.

The two outside lines then here must be D_1F_1, answering to DF, and D_2E answering to DE. D_1F_1 is then even, if outside line in the polygon corresponding to F; D_2E is even, since it has one angular point at E beside the ambiguous point and is outside line. Similarly F_2E is even being outside line in the polygon corresponding to F, and the remaining line C_2C_3 F_1F_2 is an even line having angular points only at the two ambiguous points C and F

and being inside line in both corresponding polygons. The unambiguous figure then having all its lines even is an angular circuit. Therefore the ambiguous figure can be traversed as a linear circuit, and this can be verified. For an angular traverse may begin and end at CB.

Thus to

$$C_1B^\frown BA^\frown AC_2^\frown C_2F_2^\frown F_2E^\frown ED_2^\frown$$
$$D_2C_3^\frown D_1F_1^\frown F_1C_3^\frown C_3D_1^\frown C_1B$$

in the unambiguous figure, corresponds

$$CB^\frown BA^\frown AC^\frown CF^\frown FE^\frown ED^\frown$$
$$DC^\frown DF^\frown FC^\frown CD^\frown CB$$

in the ambiguous figure (where $ED^\frown DC$ is part of the supplement of CDF, $= DC^\frown DF$; and so on).

Similarly we may traverse the ambiguous figure as a circuit from any other line.

This result is verified by Fig. 7 of § 49 which shews that Fig. 1 may be the reciprocal of a linear circuit.

Thus this ambiguous figure has the property that it can be traversed either as a circuit or as a figure with two odd lines.

It can also be traversed as a figure with four odd lines in two traverses, as shewn in Fig. 4 by an unambiguous figure.

We have seen that Fig. 1 may have for its reciprocal a figure of the type of Fig. 5,

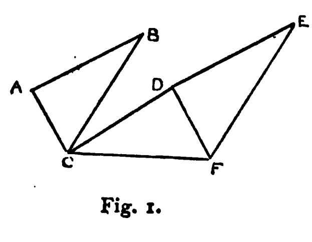


Fig. 1.

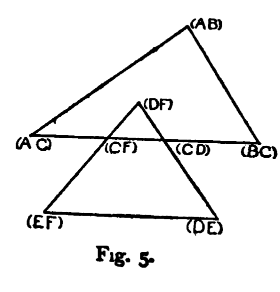
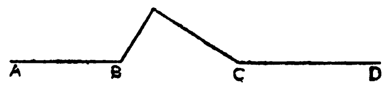


Fig. 5.

which, if all the segments are finite, is a linear circuit, whence Fig. 1 must be traversable as an angular circuit, and thus the above result is confirmed. But to every traverse of Fig. 1 must correspond one of Fig. 5. Thus to the above traverse between BC and CD, as odd lines, must correspond a traverse of Fig. 5 in which (BC) and (CD) appear as odd points. This of course can only be possible by substituting some infinite segments for finite ones. And it will be easily seen that the required traverse can be made if the traverse from AC to CF is upon the infinite segment $(AC)\_(CF)$, which necessitates that a traverse from (AC) to (BC) is on the infinite segment $(AC)\_(BC)$. Whence (BC) becomes an odd point with the finite segment $(BC)\bar{\ }(CD)$, the finite or infinite segment (BC) (AB) and the infinite segment $(BC)\_(CA)$ radiating from it. So also (CD) becomes an odd point if the traverse from (CD) to (DE) is on the infinite segment $(CD)\_(DE)$; for then there radiate from (CD) the three finite segments $(CD)\bar{\ }(DF)$ $(CD)\bar{\ }(CF)$, and $(CD)\bar{\ }(BC)$. (CF) is even if the traverse to (EF) is on $(CF)\_(EF)$.

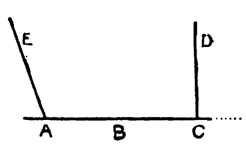
The ambiguity then of the angular figure has a corresponding ambiguity in the linear figure, and the cause of it in the linear figure, that is, the fact that two or more lines of the figure are in the same straight line, is the cause of the ambiguous points in the reciprocal angular figure.

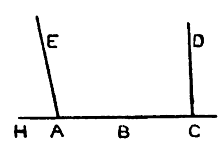
For if a given line of the figure is not in the same straight line as any other, it is clear that it makes no difference to the denomination of the points which are its extremities whether the infinite segment or the finite segment is considered to be traversed; and thus there is no ambiguity about the traverse of the figure. On the other hand supposing a line AB is in the same straight line as CD, then the infinite segment A_B includes the finite segment $C\bar{\ }D$, and vice versa.



If $A B$ and $C D$ are not adjacent, and B and C are not joined by one line of the figure; then if A_B is substituted for $A\overline{\ }B$, a line joining $B C$ is necessarily introduced as part of A_B, and thus the denomination of the point C at least is altered, and of D if there is no one line of the figure at D in the same straight line with $D C$.

If B is joined to C by one line of the figure, or if B and C coincide, then the substitution of A_B for $A\overline{\ }B$ alters the denomination of D, or, if a line is adjacent to D in the same straight line as $A D$, the denomination of the last point of the figure which is in this straight line.





Again, suppose the adjacent lines $A B$ and $B C$ are in the same straight line and no other line at C in this same straight line, and let there be at least one other line, as $C D$ at C. If A_B is substituted for $A\overline{\ }B$ another line is clearly added to the finite lines at C, and the denomination of C is changed. The denomination of A is not changed if no other line at A is in the same straight line with $A B$; but if there is another, as $A H$, A loses the line $A B$ without compensation, and its denomination is altered.

Some other examples of ambiguous figures may be added.

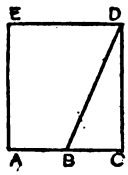


Fig. 6.

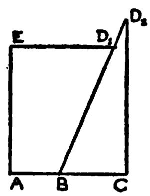


Fig. 7.

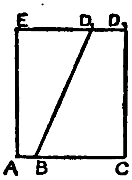


Fig. 8.

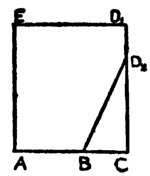


Fig. 9.

In Fig. 6, each of the lines at the ambiguous point D has one angular point on it beside D. As has been shewn, only two of these can be made even. There must therefore be one odd line at D, which may be BD as in Fig. 7, ED as in Fig. 8, or CD as in Fig. 9. The figure has one other odd line ABC.

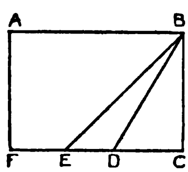


Fig. 10.

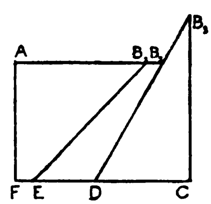


Fig. 11.

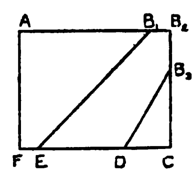


Fig. 12.

In Fig. 10 each of the lines at B has one angular point beside B. Two of them can be made even, and thus there must be two odd lines at B, which may be any two of the four at B; e.g. BA and BD as in Fig. 11; BA and BC as in Fig. 12.

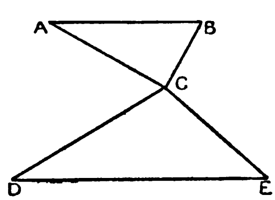


Fig. 13.

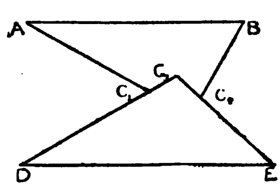


Fig. 14.

Fig. 13, which can be traversed as a finite linear circuit, cannot be traversed as an angular circuit. For it is evident that two of the lines at C must be odd lines, e. g. as in Fig. 14.

It has been already shewn (§ 63) that Fig. 13 cannot be traversed as an angular circuit, because it consists of two angular circuits, ABC and CDE, with an angular point in common (see p. 95).

Note upon the most general form of the construction of reciprocal figures.

In the foregoing, § 48, it has been said that any mode of constructing the reciprocal of a given figure may be taken which secures that if a point A is on the line corresponding to a point B the point B is on the line corresponding to the point A.

In the usual theory of reciprocation, whether by direct reference to a base conic or by the dual interpretation of a 'linear' equation, if F is a given figure and (F) its reciprocal, the rule of construction by which the line in (F) corresponding to a given point in F is found, is the same as that by which the line in F corresponding to a given point in (F) is found.

If the line (A) in (F) corresponding to a point A in F is found by the rule of construction R, this may be represented by the equation

$$(A) = R.A\,;$$

and A being determined from (A), (A) is given by a kind of inverse of the rule R, which may be expressed by the equation

$$A = R^{-1}.(A).$$

In the usual theory, if AB is a line in F, and (AB) the point corresponding to it in (F), AB is found in the same way from (AB) as the line (A) in (F) corresponding to the point A in F is found from (A); that is

$$AB = R.(AB)$$

and

$$(AB) = R^{-1}.AB.$$

Another kind of reciprocation is, however, possible in which the rule by which a line in (F) is found from a point in F is not the same as that by which a line in F is found from a point in (F).

Let the rule for finding a line in (F) from a point in F be R, and the rule for finding a line in F from a point in (F) be ρ.

Then if A is a point in F, AB a line in F, and (A) and (AB) the line and point in (F) which correspond to them,

$$\begin{aligned} (A) &= R.A \\ A &= R^{-1}(A) \\ AB &= \rho.(AB) \\ (AB) &= \rho^{-1}.AB. \end{aligned}$$

The principle of reciprocity in the usual theory may be stated thus:—

If A is upon the line corresponding to B, B is upon the line corresponding to A, or

If A is upon (B), which is $R.B$, B is upon (A), which is $R.A$.

It is a result of this that the point (AB) which corresponds to the line AB, is the intersection of the lines (A) and (B) which correspond to the points A and B respectively.

In the other kind of reciprocation where there are two rules of construction R and ρ the principle of reciprocity will be this:—

If the point A in the figure F is upon a line BC belonging to the same figure which corresponds to a point (BC) in the figure (F), then the point (BC) is upon the line (A) in the figure (F) which corresponds to the point A in figure F, and conversely. That is,

If A is upon $\rho\ (BC)$, which is BC, then (BC) is upon $R.A$, which is (A); and if (BC) is on $R.A$, A is on $\rho\ (BC)$.

Suppose now (BC) is the intersection of (B), $= R.B$, and (C), $= R.C$. By the above principle,

Since (BC) is upon $R.B$, B is upon $\rho.(BC)$;

And since (BC) is upon $R.C$, C is upon $\rho(BC)$:

Therefore $\rho(BC)$ passes through B and C.

That is the point (BC) which is the intersection of the lines (B) and (C) in (F) corresponding to the points B and C in F, corresponds to the line BC which joins these points.

Similarly if a point A in F is the intersection of the lines AB and AC in F, which correspond to the point (AB) and (AC) in (F), A corresponds to the line joining (AB) and (AC) in (F); that is (A) is the line joining (AB) and (AC).

Thus the intersection of two lines in the one figure will correspond to the line in the second figure which joins the points in the second figure corresponding to the two given lines in the first figure, as in the ordinary method of reciprocation.

If then R and ρ are so related that this principle of reciprocity holds, it is evident that if three points are collinear in one figure, the lines corresponding to them in the other figure are concurrent, and conversely.

The difference between the two methods may be formulated thus. First, or usual method:—

(i) (1) $A \succeq (A) = R.A$, and $A = R^{-1}(A)$.

(2) $AB \succeq (AB) = R^{-1}AB$, and $AB = R(AB)$.

(ii) If X is on (Y), which is $R.Y$, then Y is on (X) which is $R.X$.

(iii) Whence (AB), which is $R.AB$, is the intersection of (A), which is $R.A$, and (B), which is $R.B$.

Second method:—

(i) (1) $A \succeq (A) = R.A$; and $A = R^{-1}.A$.

(2) $AB \succeq (AB), = \rho^{-1}AB$; and $AB = \rho(AB)$.

(ii) If X is a point in F, Y a point in (F), (X) the correspondent of X in (F), and (Y) the correspondent in F of Y; so that $(X) = R.X$ and $(Y) = \rho Y$, then

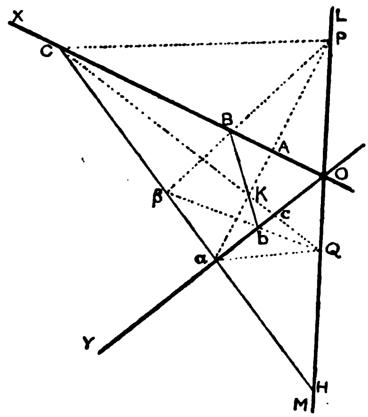
If X is upon (Y), Y is upon (X).

(iii) Whence (AB) in (F) which corresponds to AB in F, is the intersection of (A) and (B) in (F) which correspond to A and B in F.

And the point A in F which corresponds to the line (AB) (AC) in (F) is the intersection of the lines AB and AC in F which correspond to the points (AB) and (AC) in (F), i.e. the line (AB) (AC) is the line (A).

An illustration may be given which shews that R and ρ can

have the relation which is necessary to make the second method possible.



Let OX and OY be two fixed straight lines meeting in O.

Let P and Q be two fixed points on a straight line LM through O.

Draw any straight line CaH cutting OX in C and OY in a.

Join PC and Qa. Join Pa cutting OX in A; join QC cutting OY in c. Let Pa and QC intersect in K.

Draw any straight line BKb, through K cutting OX and OY in B and b respectively.

Let PB meet Ca in β.

Let Qb meet Ca in b'.

Then it may be shewn that β and b' coincide.

Join KO.

Then, from the ranges determined by the pencil at K upon OX and OY, we have

$$\{OABC\} = \{Oabc\}$$

Hence, taking the anharmonic ranges determined by the pencils $P\{OABC\}$ and $Q\{Oabc\}$ upon the straight line CaH, we have

$$\{Ha\beta C\} = \{Hab'C\}$$

whence b' coincides with β.

Let the rule R for finding in (F) the correspondent Bb of a point β in F be this :—

Join βP intersecting OX in B, and βQ intersecting OY in b. The straight line Bb in (F) is the correspondent of β in F.

Let the rule ρ for finding in F the correspondent Ca of the point K in (F) be this:—

Join KP cutting OY in a, and KQ cutting OX in C.

Ca in F is the correspondent of K in (F).

The theorem proved above shews that the correspondent in (F) of any point β upon the straight line Ca in F passes through K which is the correspondent in (F) of Ca in F.

It also follows that, if any point K' is taken upon a straight line Bb in (F), the line in F which corresponds to K' passes through β, the point in F which corresponds to Bb in (F).

Thus if any point β in F is upon a line in F which corresponds to a point K in (F), K is upon the line in (F) which corresponds to β in F: and if any point K in (F) is upon a line in (F) which corresponds to a point β in F, β is upon a line in F which corresponds to K in (F).

Thus R and ρ fulfil the required conditions, the figures F and (F) are reciprocal, and the intersection of any two straight lines in the one figure corresponds to the line joining the points which correspond to these two lines in the other figure.

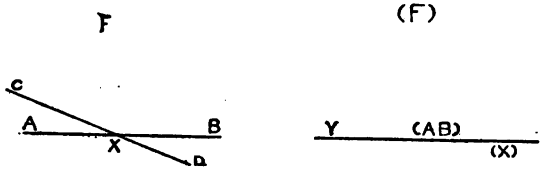
Since to every point in a given straight line Ca in F there corresponds a straight line through K, the point in (F) corresponding to Ca, it is evident that to the linear traverse of the line Ca in F corresponds the angular traverse of the angle at K in (F) between the lines at K corresponding to C and a, which are Cc and aA respectively.

Thus the theory of the correspondence of linear and angular traverses is applicable to reciprocal figures constructed as in this case, and, in general, to all reciprocal figures constructed according to the second method where R and ρ fulfil the above described conditions.

The construction of the above illustration may be generalized.

Let R be any rule of construction such that the correspondent in (F) of any point upon any given straight line in F passes through one point K fixed relatively to the given line, and, conversely, every line in (F) through the latter point has its R-correspondent in F upon the given straight line in F; then

this is enough for the required correspondence, however K may be fixed. That is if ρ^{-1} represents the rule by which the point K is fixed in relation to the given line (and therefore ρ the rule by which the given line might be determined from the point), R and ρ fulfil the required conditions, and F and (F) are reciprocal figures.



For suppose a line AB in F has corresponding to it by the rule ρ^{-1} in (F) the point (AB).

Let X be a point in AB; then, by hypothesis, if (X) is the line in (F) corresponding by the rule R to X in F, (X) passes through (AB).

Upon (X) take any point Y in (F), and let CD be the line in F determined by the rule ρ from Y.

By hypothesis every line through Y in (F) has a correspondent point in F which lies upon CD, the line being determined from its correspondent point by the rule R. But (X) passes through Y. Therefore the correspondent of (X) lies upon CD. That is CD passes through X.

Similarly the correspondent of any point on (X) determined by the rule ρ passes through the point X.

Thus the line corresponding, by the rule ρ, to any point in a given straight line in (F) passes through a point in F fixed in relation to the given line by the rule R^{-1}.

As a similar law, by hypothesis, holds for the correspondents by the rule R of all points in a given line of F, it follows that if a point in one figure is on the correspondent in that figure to a point in the other, the second point is upon the first point's correspondent in the second figure.

Whence, as shewn above (p. 121–2), the figures are reciprocal.

It is clear that the method, generalized as above, in which R and ρ are different is wider than the usual method, and includes it as the particular case in which R and ρ are identical.

It is also the most general rule for the construction of figures for which theorems can be reciprocated in the usual way, that is by the interchange of 'line' and 'point,' &c.

For it is a necessary condition of all such reciprocation that the correspondents according to a rule R of all points on any given straight line in the one figure should pass through a point in the other fixed relatively to the given straight line, and conversely each line through the latter point should have its R^{-1}-correspondent on the given line. And it has been shewn above that this condition is also the sufficient condition, inasmuch as the resulting rule ρ by which the point is fixed, whether identical with R or not, must satisfy the conditions of the kind of reciprocation in question.

Appendix. Angular Coordinates and Vectors.

§ 72. *Angular coordinates of a line.*

The familiar theory of reciprocation has advanced from the idea of the coordinates of a point and equation to a straight line to that of the equation to a point and the coordinates of a line as their correspondents. But the same system of rectilinear coordinates is presupposed in both cases.

For complete correspondence we seem to require that a reciprocation should be made of the coordinate system itself. This has not been done; and, indeed, the ordinary correspondence merely between line and point does not suggest it. To correspond to 'line coordinates' we cannot have 'point coordinates,' for a mere point cannot be a 'coordinate.' Thus we could not have corresponding to the rectilinear coordinates of a point and the rectilinear (coordinate) equation to a line, the point coordinates of a line, and the point equation to a point.

But linear coordinates are lines of definite length, and, as we have seen, what corresponds to a line of definite length, is not a point, but an angle at a point. Thus the reciprocal of a linear coordinate would be an angular coordinate.

We have therefore to seek, as corresponding to the

rectilinear coordinates of a point and the rectilinear (coordinate) equation to a straight line, the angular coordinates of a straight line and the angular equation to a point.

The required correspondence can be got as follows:—

In the usual coordinate system,

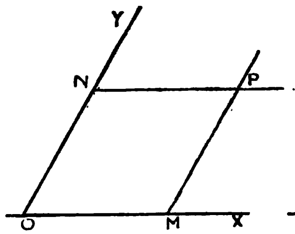


Fig. 1.

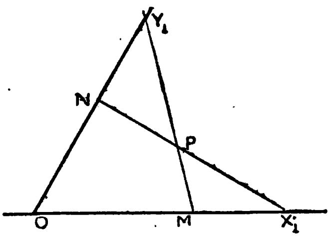


Fig. 2.

if OX and OY are axes of coordinates, and PM (or ON) and PN (or OM) are the coordinates of a point P, the position of P may be regarded as determined thus:— Let a straight line move from coincidence with OY, always remaining parallel to OY to a position PM, and let a line move from OX always parallel to OX to a position PN; P is determined as the intersection of these positions.

Now the movement of the line from OY to PM is the movement of an angular traverse from OY when the vertex of the angle traversed is the 'point at infinity' upon OY; and the movement of the line from OX to PN is the movement of an angular traverse from OX when the vertex of the angle traversed is the 'point at infinity' upon OX.

Thus Fig. 1 is a case of Fig. 2 where the vertices of the angles traversed are Y_1 on OY at a finite distance from O, and X_1 on OX at a finite distance from O.

The points X_1 and Y_1 may be conveniently called 'the director points.'

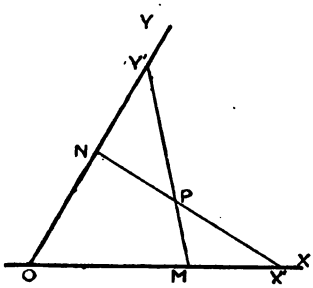


Fig. 2.

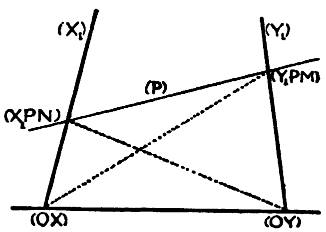


Fig. 3.

The reciprocal of this may be found thus:

Let O be a fixed point:

Let OX and OY be fixed lines through it.

Let X_1 be a fixed point on OX,

and Y_1 be a fixed point on OY.

Let an angular traverse pass from XO through the point X_1 to a position X_1PN,

and an angular traverse from YO through the point Y_1 to a position Y_1PM.

P is the intersection of X_1PN and Y_1PM.

Then if M is the intersection of Y_1PN and OX,

and N the intersection of X_1PN and OY,

the lines OM and ON are the linear coordinates of the point P.

Let (O) be a fixed line:

Let (OX) and (OY) be fixed points on it.

Let (X_1) be a fixed line through (OX),

and (Y_1) a fixed line through (OY).

Let a linear traverse pass from (OX) along (X_1) to a position (X_1PN),

and a linear traverse from (YO) along the line (Y_1) to a position (Y_1PM).

(P) is the line joining (X_1PN) and (Y_1PM).

Then if (M) is the line joining (Y_1PN) and (OX),

and (N) the line joining (X_1PN) and (OY),

the angles $(O)\widehat{\ }(M)$ and $(O)\widehat{\ }(N)$ are the angular coordinates of the line (P).

The line (O) may be called 'the origin of coordinates,'

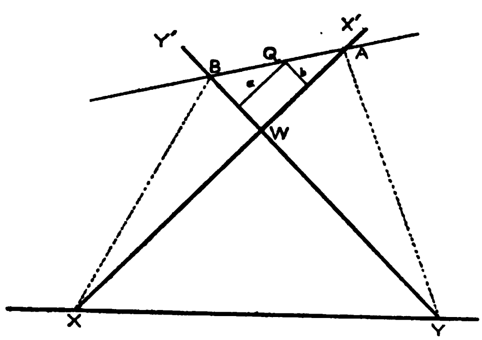
as being the line from which the angular coordinates are measured. (OX) and (OY) may be called 'the vertices of coordinates.' (X_1) and (Y_1) may be called 'the directors.'

§ 73. *Angular equation to a point.*

The equation to a point is the equation which must be satisfied by the coordinates of any line which passes through it.

Let the straight line (P) pass through a fixed point (PR). Then if the coordinates $(O)\widehat{\ }(M)$ and $(O)\widehat{\ }(N)$ of the line (P) are Θ and Φ respectively, an equation may be found connecting trigonometrical functions of Θ and Φ with known constants; and if Θ and Φ are made variable, this will be the angular equation to the point (PR).

With a figure so general as that of Fig. 3, § 72, the equation has a cumbrous form. It may be simplified by a modification of the figure.



Let the points X and Y be the vertices of angular coordinates, then the straight line XY is the origin of coordinates. Let the directors XX' and YY' make each half a right angle with XY, and let W be the point where they intersect at right angles.

Let the straight line BA, which passes through a fixed point Q, meet the directors in the points A and B respectively. Its angular coordinates are the angle BXY and the angle AYX. Let $BXY=\Phi$, and $AYX=\Theta$.

Let the positive direction of rotation for Φ be from XY towards XW, and for Θ from XY towards YW.

Let the perpendiculars from Q upon XX' and YY' be b and a respectively; and let $WX=c=WY$.

$$\frac{a}{WA}+\frac{b}{WB}=1,$$

$$WA=c\tan AYW=c\tan\left(\Theta-\frac{\pi}{4}\right),\quad WB=c\tan\left(\Phi-\frac{\pi}{4}\right),$$

whence

$$a\cot\left(\Theta-\frac{\pi}{4}\right)+b\cot\left(\Phi-\frac{\pi}{4}\right)=c.$$

If now $QW=r$, and the angle $QWA=\alpha$, and if the positive direction of rotation for α corresponds to that for Φ; that is, the rotation is from WA towards WB, the angular equation to the point Q is

$$r\cos\alpha\,.\cot\left(\Theta-\frac{\pi}{4}\right)+r\sin\alpha\cot\left(\Phi-\frac{\pi}{4}\right)=c.$$

If we determine the position of the point Q by the angles QXY $(=\phi_1)$, and QYX $(=\theta_1)$, the equation may be put in a form which contains only angles starting from the origin XY, and at the vertices of coordinates X and Y.

$$a=QY\sin\left(\theta_1-\frac{\pi}{4}\right),$$

$$QY=XY\cdot\frac{\sin\phi_1}{\sin(\theta_1+\phi_1)},$$

and these are corresponding values for b.

Also $$XY=c\sqrt{2}.$$

Hence the equation

$$a\cot\left(\Theta-\frac{\pi}{4}\right)+b\cot\left(\Phi-\frac{\pi}{4}\right)=c$$

becomes

$$\left\{\sin\phi_1\sin\left(\theta_1-\frac{\pi}{4}\right)\right\}\cot\left(\Theta-\frac{\pi}{4}\right)$$
$$+\left\{\sin\theta_1\sin\left(\phi_1-\frac{\pi}{4}\right)\right\}\cot\left(\Phi-\frac{\pi}{4}\right)=\frac{\sin(\theta_1+\phi_1)}{\sqrt{2}}.$$

In this the direction of rotation for the angles θ_1 and ϕ_1 is the same as that for Θ and Φ respectively.

A simpler form is given in the next section.

§ 74. *Angular coordinates of a point. Angular equation to a line.*

If we call θ_1 and ϕ_1 the coordinates of Q, and in general the angles which lines drawn from any point to the vertices of coordinates make with the line XY (the origin of coordinates) the coordinates of that point, we may find an equation to a straight line in the form of a condition which the angular coordinates of any point upon it must satisfy.

From the equation to the point (θ_1, ϕ_1), viz.:

$$\sin\phi_1\sin\left(\theta_1-\frac{\pi}{4}\right)\cot\left(\Theta-\frac{\pi}{4}\right)+\sin\theta_1\sin\left(\phi_1-\frac{\pi}{4}\right)\cot\left(\Phi-\frac{\pi}{4}\right)$$
$$=\sin\frac{(\theta_1+\phi_1)}{\sqrt{2}},$$

we get as condition of collinearity of three points (θ', ϕ'), (θ_1, ϕ_1), (θ_2, ϕ_2) the equation

$$\begin{vmatrix} \sin\phi'\sin\left(\theta'-\frac{\pi}{4}\right) & \sin\theta'\sin\left(\phi'-\frac{\pi}{4}\right) & -\sin\frac{(\theta'+\phi')}{\sqrt{2}} \\ \sin\phi_1\sin\left(\theta_1-\frac{\pi}{4}\right) & \sin\theta_1\sin\left(\phi_1-\frac{\pi}{4}\right) & -\sin\frac{(\theta_1+\phi_1)}{\sqrt{2}} \\ \sin\phi_2\sin\left(\theta_2-\frac{\pi}{4}\right) & \sin\theta_2\sin\left(\phi_2-\frac{\pi}{4}\right) & -\sin\frac{(\theta_2+\phi_2)}{\sqrt{2}} \end{vmatrix}=0.$$

If then θ' and ϕ' are variables, the equation is the equation

of a straight line passing through the two points (θ_1, ϕ_1) and (θ_2, ϕ_2): and we may write now θ for θ' and ϕ for ϕ'.

Expanding $\sin\left(\theta - \frac{\pi}{4}\right)$, &c., we get for the first row of the determinant

$$\sin\phi\,(\sin\theta - \cos\theta) \quad \sin\theta\,(\sin\phi - \cos\phi)$$
$$-\sin\theta\cos\phi - \cos\theta\sin\phi,$$

and similar expressions for the other rows.

To the first column add the second, and subtract the third. Divide by 2, and from the second column of the resulting determinant subtract the first column. In the resulting determinant from the third column subtract the second. The equation thus becomes

$$\begin{vmatrix} \sin\theta\sin\phi & -\sin\theta\cos\phi & -\cos\theta\sin\phi \\ \sin\theta_1\sin\phi_1 & -\sin\theta_1\cos\phi_1 & -\cos\theta_1\sin\phi_1 \\ \sin\theta_2\sin\phi_2 & -\sin\theta_2\cos\phi_2 & -\cos\theta_2\sin\phi_2 \end{vmatrix} = 0.$$

Dividing the first row by $\sin\theta\sin\phi$, the second by $\sin\theta_1\cos\phi_1$, and the third by $\sin\theta_2\sin\phi_2$, this becomes

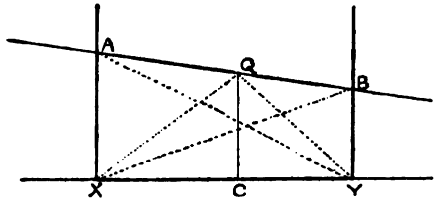
$$\begin{vmatrix} 1 & -\cot\phi & -\cot\theta \\ 1 & -\cot\phi_1 & -\cot\theta_1 \\ 1 & -\cot\phi_2 & -\cot\theta_2 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} \cot\phi & \cot\theta & -1 \\ \cot\phi_1 & \cot\theta_1 & -1 \\ \cot\phi_2 & \cot\theta_2 & -1 \end{vmatrix} = 0.$$

Thus a result which of course can be got without reference to 'directors' is connected with the 'director equation.' It also may be got immediately from this equation if we simplify the figure further by taking directors perpendicular to *X Y*, the origin of angular coordinates. This modification was not made in the foregoing, because the position of a line at right angles to the origin could not then be determined by two angular

coordinates. It yields, however, a form of director equation convenient for certain purposes.



Let Q be a fixed point (ϕ_1, θ_1) through which passes a variable straight line AB, cutting in A and B directors perpendicular to XY. $BXY = \Phi$, and $AYX = \Theta$. If QC is perpendicular to XY, it may be easily shewn that

$$XC \tan\Phi + CY \tan\Theta = QC.$$

Whence we get as angular equation to the point (ϕ_1, θ_1)

$$\cot\phi_1 \tan\Phi + \cot\theta_1 \tan\Theta = 1.$$

Hence the condition of collinearity of three points (θ, ϕ), (θ_1, ϕ_1), (θ_2, ϕ_2) is

$$\begin{vmatrix} \cot\phi & \cot\theta & -1 \\ \cot\phi_1 & \cot\theta_1 & -1 \\ \cot\phi_2 & \cot\theta_2 & -1 \end{vmatrix} = 0.$$

And if θ and ϕ are variable this is the equation to a straight line passing through the fixed points (θ_1, ϕ_1) and (θ_2, ϕ_2).

§ 75. *Illustrations in focal conics.*

The method of determining the position of a point by angular coordinates corresponds, in the system of Cartesian or rectilinear coordinates, to the method of determining the position of a straight line by its intercepts on the axis considered as its coordinates.

The indeterminateness of the former method, when the point is upon the line which is the origin of angular coordinates, corresponds to the indeterminateness of the

latter method when the line passes through the point which is the origin of rectilinear coordinates.

The indeterminateness of the above equation to a straight line, in terms of angular coordinates of points, when applied to the line which is the origin of angular coordinates, corresponds to the indeterminateness which the equation to a point in Cartesian coordinates, in terms of the coordinates of straight lines (as above), assumes, when the point is origin of rectilinear coordinates.

When in an angular equation the value of one of the coordinates is to be determined which corresponds to a given value of the other, it must be remembered that *θ and ϕ are always of the same sign, and that, in magnitude,* $(\theta+\phi) < \pi$.

The equation to the parabola in terms of angular coordinates of points, the vertices of coordinates being the focus and the intersection of the axis with the directrix is

$$\tan\phi = \sin\theta$$

where θ is the angle at the focus.

The equation to the ellipse, the vertices of coordinates being the foci, is

$$\sin\theta + \sin\phi = \frac{a}{c}\sin(\theta+\phi),$$

where a is the length of the major axis, and c the distance between the foci.

The equation to the hyperbola, the vertices of coordinates being the foci, is (remembering the above convention)

$$\sin\theta - \sin\phi = \pm\frac{a}{c}\sin(\theta+\phi).$$

The last two equations shew at once the symmetry of the curves with respect to the major and minor axes.

The equation to the parabola shews the symmetry of the curve with respect to its axis, and that the lines joining the focus to the extremities of a chord which passes through

the other vertex of coordinates make equal angles with the axis.

The equation to the ellipse, if the same vertices of coordinates are taken as for the parabola, is

$$\tan\phi = e\sin\theta,$$

and the equation to the hyperbola is

$$\tan\phi = \pm e\sin\theta.$$

These equations shew that the ellipse and hyperbola have the property above stated for the parabola.

We have found as the equation to a point the equation to be satisfied by the two angular coordinates of every line passing through it.

Similarly we may find as the equation to a curve the equation to be satisfied by the two angular coordinates of every tangent to it. This may be illustrated in the case of the parabola.

Let the vertices of coordinates be the focus and the intersection of the axis with the directrix.

Let the 'directors' be the latus rectum produced and the directrix.

Let the angular coordinates of a point on the curve be θ_1 and ϕ_1, and let the angular coordinates of a line passing through the point $(\theta_1\,\phi_1)$ be Θ and Φ.

The angular equation to the point $(\theta_1\,\phi_1)$ is (§ 74 *fin.*)

$$\cot\phi_1\tan\Phi + \cot\theta_1\tan\Theta = 1.$$

But in the parabola

$$\tan\phi_1 = \sin\theta_1.$$

Whence $$\tan\Phi + \cos\theta_1\tan\Theta = \sin\theta_1.$$

Solving for $\sin\theta_1$ we get

$$\sin\theta_1 = \frac{\tan\Phi \pm \tan\Theta\sqrt{1-\tan^2\Phi+\tan^2\Theta}}{1+\tan^2\Theta}.$$

These two values of $\sin\theta_1$ are clearly the values of the sines of the θ-coordinates of the two points in which the line $(\phi,\ \theta)$ meets the curve. Hence if this line is the

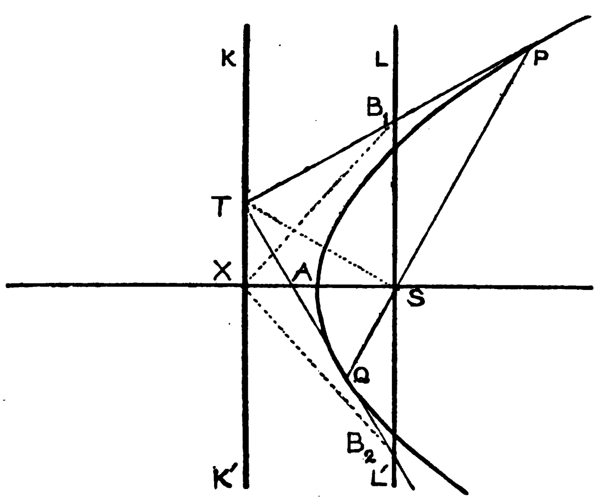
tangent at (θ_1, ϕ_1) these two values must coincide, and the expression under the root must vanish.

Thus $$1-\tan^2\Phi+\tan^2\Theta=0$$
$$\therefore\ \cot^2\Phi=\cos^2\Theta$$
$$\therefore\ \cot\Phi=\pm\cos\Theta$$

which is the required equation to the parabola.

It is evident that the absolute magnitudes of Φ and Θ, whether negative or positive, cannot be greater than $\frac{\pi}{2}$.

The equation $\cot\Phi=\pm\cos\Theta$ shews that to a given value of Θ correspond two values of Φ equal in magnitude and opposite in sign. For $\cot\Phi$ may be positive or negative, and the angle greater than $\frac{\pi}{2}$ which has a given value of the cotangent is excluded. This gives at once the following theorem.



Let S be the focus; X the intersection of the axis SAX with the directrix KK': TP and TQ the tangents at P and Q from a point T on the directrix: $L'SL$ the perpendicular to the axis through the focus.

Let S and X be vertices of angular coordinates: KK' and LL' the 'directors.'

Let TP and TQ meet LL' in B_1 and B_2 respectively.

The angular coordinates of TP are the angle B_1XS, $= \Phi'$, and the angle TSX, $= \Theta'$.

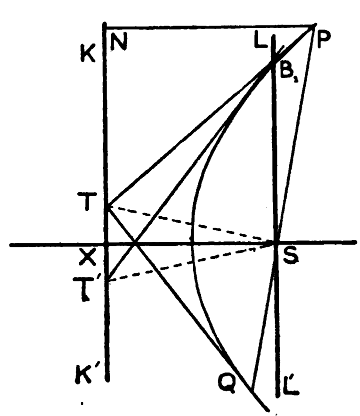
The coordinates of TQ are B_2XS, $= \Phi''$, and Θ'. The two values of Φ which correspond to $\Theta = \Theta'$ in the equation $\cot \Phi = \pm \cos \Theta$ are Φ' and Φ''.

Hence, by the above, $\Phi' = \Phi''$. That is, the angle $B_1XS =$ the angle B_2XS. Hence $SB_1 = SB_2$.

Thus if from a point in the directrix a pair of tangents are drawn to the parabola, these tangents intersect the latus rectum produced in two points which are equidistant from the focus.

Again it follows from the above equation between the coordinates of the tangent, that to a given value of Φ correspond two values of Θ equal in magnitude and opposite in sign. For to a given value of the cosine of Θ correspond a positive and negative angle which are equal to one another, while to a given value of $\cot \Phi$ in the equation $\cos \Theta = \pm \cot \Phi$ only one value of $\cos \Theta$ can correspond, since $\cos \Theta$ must always be positive. (N.B. The positive direction of rotation for Φ is the negative direction for Θ.)

This leads to the following theorem.



Let the tangent at P cut LL' in B_1, and KK' in T.

Let the other tangent from B_1 meet the curve in Q and cut KK' in T'.

The coordinates of TP are the angle B_1XS, $=\Phi'$, and the angle TSX, $=\Theta'$. The coordinates of B_1T' are the angle B_1XS, $=\Phi'$, and the angle $T'SX=\Theta''$.

Thus the two values of Θ which correspond to the value Φ' for Φ are Θ' and Θ''.

Hence, by the above, these angles are equal, that is the angle $TSX=$ the angle $T'SX$.

Hence $TX=T'X$, and $ST=ST'$.

Thus, if from any point on the latus rectum produced a pair of tangents be drawn to the parabola, these tangents intersect the directrix in two points equidistant from the intersection of the axis and directrix, and equidistant from the focus.

Also $$\sin\theta_1=\frac{\tan\Phi\pm\tan\Theta\sqrt{1-\tan^2\Phi+\tan^2\Theta}}{1+\tan^2\Theta},$$
whence, as the expression under the root vanishes for the tangent,
$$\sin\theta_1=\frac{\tan\Phi}{1+\tan^2\Theta}=\tan\Phi\,.\cos^2\Theta.$$
But $$\cot^2\Phi=\cos^2\Theta,$$
$$\therefore\ \sin\theta_1=\cot\Phi=\pm\cos\Theta.$$
In the figure $\theta_1=PSX$, and $\Theta=TSX$.

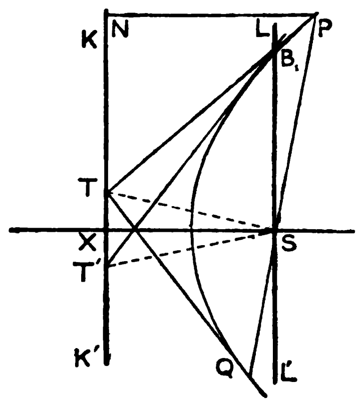
Thus $\cos TSX=\sin PSX$.

Whence TSP is a right angle.

Similarly if TQ is the other tangent from T, TSQ is a right angle.

Thus QSP is a straight line.

Hence tangents at the extremity of a focal chord intersect in the directrix, and the line joining their point of intersection to the focus is perpendicular to the focal chord.



Let PN be the perpendicular from P upon the directrix. $SP = PN$ is the property of the parabola from which the equation $\tan \phi_1 = \sin \theta_1$ was deduced.

Since TSP is a right angle,

$$\sin TPS = \frac{SP}{PT},$$

also $$\sin TPN = \frac{PN}{PT} = \frac{SP}{PT} = \sin TPS.$$

Hence the angle $TPN =$ the angle TPS.

Thus the tangent at a point makes equal angles with the focal distance of the point and the perpendicular from the point to the directrix.

Hence it is easily shewn that the angle PTQ is a right angle. Thus the tangents at the extremity of a focal chord intersect at right angles in the directrix.

Let PT and TQ meet LL' in B_1 and B_2, and let the other tangent from B_1 cut the directrix in T'.

Since B_1TB_2 is a right angle and $SB_1 = SB_2$,

$$ST = SB_1 = SB_2.$$

But $ST' = ST$, as above shewn.

$\therefore$ the points B_1, B_2, T' and T lie on the circumference of a circle of which S is the centre.

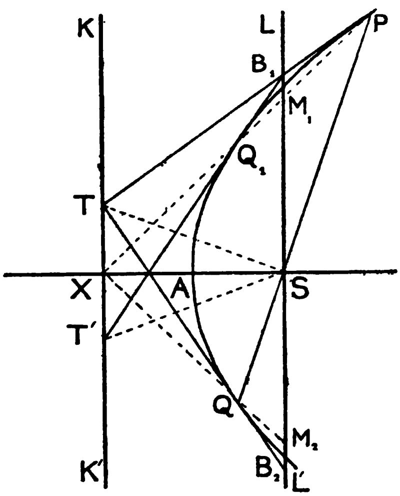
This gives a simple construction for the tangents from a point in the directrix or in the latus rectum produced.

Again $\qquad \sin\theta_1 = \cot\Phi,$

and $\qquad \tan\phi_1 = \sin\theta_1,$

$$\therefore \tan\phi_1 = \cot\Phi.$$

Hence the following theorem



Let XP intersect LL' in M_1.

$$BXS = \Phi. \qquad TSX = \Theta.$$
$$PXS = \phi_1. \qquad PSX = \theta_1.$$

Then, the angles having their proper signs,

$$PSX = \theta_1 = \tfrac{\pi}{2} + \Theta.$$

By the above $\tan\phi_1 = \tan PXS - \cot\Phi = \cot B_1XS$

$$\therefore B_1XS = \tfrac{\pi}{2} - PXS = SM_1X.$$

Hence the triangle B_1SX and XSM_1 are similar.

$$\therefore SM_1 . SB_1 = SX^2$$

Hence SX touches the circle described about the triangle B_1M_1X.

Similarly if the other tangent from T meets the curve in Q and LL' in B_2, and if XQ cuts LL' in N,

$$SM_2 . SB_2 = SX^2 = SM_1 . SB_1.$$

But $$SB_1 = SB_2 \therefore SM_2 = SM_1.$$

Hence the angle $PXS =$ the angle QXS. This last proposition may be also got as shewn below for the central focal conics.

The following relations between the angles of the figure, Θ, Φ, θ_1, and ϕ_1 having their proper signs, may be easily verified:

$$XPS = \Phi - \Theta,$$

$$TPS = \tfrac{1}{2}(\pi - \theta_1) = \tfrac{1}{2}\left(\tfrac{\pi}{2} - \Theta\right) = \tfrac{1}{2} B_1ST,$$

$$SB_1T = PTS \, (\text{since } SB_1 = ST) = \tfrac{1}{2}\left(\tfrac{\pi}{2} + \Theta\right) = \tfrac{\theta_1}{2} = \tfrac{1}{2} PSX.$$

If the ellipse is referred to one focus and the intersection of its major axis with the directrix as vertices of coordinates, the directors being the latus rectum produced and the corresponding directrix, then the equation to it in terms of the coordinates of points is

$$\tan\phi = e\sin\theta.$$

The equation to the ellipse in terms of the coordinates of tangents may be shewn, by the method employed in the case of the parabola, to be

$$e^2\cot^2\Phi = \cos^2\Theta.$$

And similarly for the hyperbola. Thus, as in the parabola, to a given value of Θ correspond two values of Φ, equal in magnitude and opposite in sign, and to a given value of Φ correspond two values of Θ, equal in magnitude and opposite in sign.

Hence it follows immediately that the tangents from a point in the directrix cut the corresponding latus rectum produced in two points equidistant from the axis; and that the tangents from a point in the latus rectum produced cut the corresponding directrix in two points equidistant from the axis.

This reciprocal relation, then, between latus rectum and corresponding directrix holds for all focal conics.

Also, if θ_1 and ϕ_1 are the angular coordinates of the point of contact, it may be shewn easily that

$$\sin\theta_1 = \frac{\tan\Phi . \cos^2\Theta}{e}.$$

But $$\cos^2\Theta = e^2\cot^2\Phi,$$

$$\therefore\ \sin\theta_1 = e\cot\Phi = \pm\cos\Theta.$$

From the equation

$$\sin\theta_1 = \pm\cos\Theta$$

it follows that in the figure for the ellipse or hyperbola, as in that for the parabola, that

$$\Theta - \theta_1 = \pm\tfrac{\pi}{2},$$

where the angles have their proper signs.

But $\Theta \backsim \theta_1$ is the angle between the focal distance of the point of contact and the line joining the focus to the point where the tangent intersects the directrix.

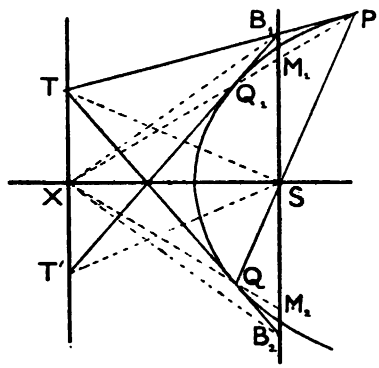
Hence it follows that the tangents at the extremity of a focal chord intersect in the directrix, and that the line joining their intersection to the focus is perpendicular to the focal chord.

Also $$\tan\phi_1 = e\sin\theta_1 = e^2\cot\Phi.$$

Whence if B_1 and B_2 be the points in which the tangents from a point in the directrix cut the latus rectum produced, and M_1 and M_2 are the points where the line joining the points of contact to the intersection, X, of the axis with the directrix cut the latus rectum,

$$SM_1 . SB_1 = e^2 SX^2 = SM_2 SB_2$$

$\therefore$, since $SB_1 = SB_2$, $SM_1 = SM_2$



Hence the angle $PXS =$ the angle QXS.

Thus if the coordinates of the tangent TQ are Θ and Φ' and those of the point Q, θ_1' and ϕ_1',

$$\Phi' = \Phi, \text{ and } \phi_1' = \phi_1.$$

This may also be deduced simply thus:—

Θ being the same for both tangents,

$$e \cot \Phi = \pm \cos \Theta$$
$$e \cot \Phi' = \pm \cos \Theta$$

$\therefore$ (as the figure shews) Φ and Φ' are equal in magnitude.

Also
$$\tan \phi_1 = e^2 \cot \Phi,$$
$$\tan \phi_1' = e^2 \cot \Phi',$$
$$\therefore \tan \phi_1 = \tan \phi_1',$$
$$\therefore \phi_1 = \phi_1'.$$

That is, the angle $PXS =$ the angle QXS.

Thus in any focal conic the lines joining the extremities of a focal chord to the intersection of the axis and directrix make equal angles with the axis. This also follows directly from the 1st, 4th, and 5th equations of this section.

Also PX cuts the curve in Q_1, the point of contact of the tangent from B_1; and the tangents from B_1 and Q intersect on the axis.

If the directors cut one another, as in § 73, to find the angular tangential equation, we can obviously use the Cartesian (point-coordinate) equation to the curve referred to the directors as axes, to eliminate a or b from the equation

$$a \cot \Theta' + b \cot \Phi' = c,$$

$$\text{where } \Theta' = \Theta - \tfrac{\pi}{4}, \text{ and } \Phi' = \Phi - \tfrac{\pi}{4}.$$

If the angle ω between the directors is oblique, a and b being drawn parallel to the directors, and the directors making equal angles with the origin, the above equation becomes

$$a \frac{\sin(\omega - \Theta')}{\sin \Theta'} + b \frac{\sin(\omega - \Phi')}{\sin \Phi'} = c.$$

For example, if the parabola is referred to its axis and the tangent at the vertex as directors, and if d is the focal distance of the vertex, its equation, in terms of coordinates of tangents, is

$$\tan \Theta \left(\cot^2 \Phi \tan \Theta + \frac{c}{d}\right) = 0,$$

and therefore for all tangents except that at the vertex

$$\cot^2 \Phi \tan \Theta + \frac{c}{d} = 0.$$

Similarly if the hyperbola is referred to its asymptotes as directors, and $ab = k^2$ is the Cartesian equation of it referred to the directors as axes, then, ω being the angle between the asymptotes, the equation between the coordinates of all tangents except the asymptotes is

$$\frac{\sin(\omega - \Phi')}{\sin \Phi'} \cdot \frac{\sin(\omega - \Theta')}{\sin \Theta'} = \frac{c^2}{4k^2}.$$

When it follows at once that the intercept of the tangent between the asymptotes is bisected at the point of contact.

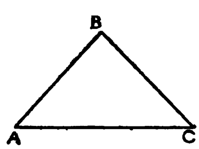
The equation to a curve in the form of a condition to be satisfied by the coordinates of its points, whether angular or rectilinear, corresponds to the linear traverse of the curve. The equation to a curve in the form of a condition to be satisfied by the coordinates of its tangents corresponds to the angular traverse.

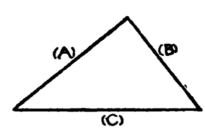
§ 76. *Angular vectors.*

The conception of angular traverses suggests a new kind of vectors, which may be called Angular Vectors.

If the traverses of two paths, each of which leads from a given point A to another point C, are considered 'equivalent,' as having the same result, that is, of changing the position from A to C, and are therefore equated, we have the ordinary equation of vectors; for a vector is the traverse of a straight line of finite length.

Thus in the triangle ABC





$A^{-}B^{-}C = A^{-}C$ is the equation of linear traverses from A to C, and if this is written $A^{-}B + B^{-}C = A^{-}C$, we have the vector equation.

If proceeding by one path we return by the other there is no change of position, and this may be expressed by the equation

$$A^{-}B^{-}C^{-}A = 0, \text{ or } A^{-}B + B^{-}C + C^{-}A = 0.$$

Thus the traverse of any circuit would be equated to zero, which is a way of interpreting the zero value for the number of traverses of a circuit in § 40 above.

On exactly the same principle we may equate the angular traverses, each of which brings a line from a given position (A) to another position (C).

Thus for the triangular circuit (A) (B) (C) we have

$$(A)\widehat{\ }(B)\widehat{\ }(C) = (A)\widehat{\ }(C), \text{ or } (A)\widehat{\ }(B)\widehat{\ }(C)\widehat{\ }(A) = 0,$$

which may be written

$$(A)\widehat{\ }(B) + (B)\widehat{\ }(C) = (A)\widehat{\ }(C),$$
$$\text{or } (A)\widehat{\ }(B) + (B)\widehat{\ }(C) + (C)\widehat{\ }(A) = 0,$$

as fundamental equation for angular vectors, just as

$$A\overline{\ }B + B\overline{\ }C = A\overline{\ }C, \text{ or } A\overline{\ }B + B\overline{\ }C + C\overline{\ }A = 0,$$

is the fundamental equation for linear vectors.

Now in the equation of angular vectors we may take either the upper or the lower angular sign for the angle between any two of the lines, e.g.

$$(A)\smile(B) + (B)\widehat{\ }(C) = (A)\widehat{\ }(C)$$
$$(A)\smile(B) + (B)\smile(C) = (A)\widehat{\ }(C)$$

&c., &c., there being eight equations in all.

Similarly if infinite traverses are admitted we may substitute an infinite segment for any finite segment, e.g.

$$A\_B + B\overline{\ }C = A\_C,$$

and there will be eight such equations in all.

If the equation of linear vectors is formulated in the proposition that any two sides of a triangle are 'equivalent' to the third, the corresponding statement for angular vectors will be that any two angles of a triangle are 'equivalent' to the third.

In the case of angular vectors, if AB arrives at coincidence with AC by revolving through the angle BAC, A remaining fixed on both lines, then the point B is on the same side of A as C is.

Suppose that the angular traverses from AB to BC, and from BC to AC, which are equivalent to the traverse from AB to AC, are to have as their result

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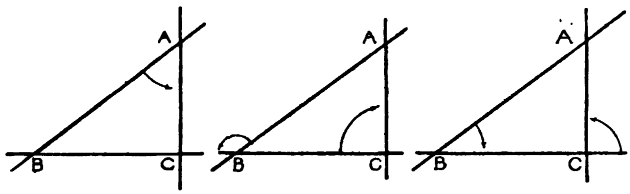


not only that AB coincides with AC, but also that the new position of B upon AC is to be on the same side of A as C is, then the revolutions at B and C must be through an exterior angle at B and an interior angle at C, or vice versa. And if we give opposite signs to revolutions towards opposite sides of the same line, we arrive at the result that the exterior angle of a triangle less the interior angle is equivalent to the other interior angle: or, the exterior angle of a triangle is 'equivalent' to the two interior and opposite angles, which of course is not the same as the proposition that the exterior angle is equal in magnitude to the two interior and opposite angles.

§ 77. *Composition of angular vectors.*

Vector equations such as

$A^{-}B^{-}C = A^{-}D^{-}C$, and $(A)^{\frown}(B)^{\frown}(C) = (A)^{\frown}(D)^{\frown}(C)$,

or $$A^{-}B + B^{-}C = A^{-}D + D^{-}C,$$

and $$(A)^{\frown}(B) + (B)^{\frown}(C) = (A)^{\frown}(D) + (D)^{\frown}(C),$$

may be conveniently written

$$A^{-}B\,B^{-}C = A^{-}D\,D^{-}C,$$

and $$(A)^{\frown}(B)\,(B)^{\frown}(C) = (A)^{\frown}(D)\,(D)^{\frown}(C).$$

These represent the composition of conterminous segments or conterminous angles.

In them a segment or angle may be transferred from one side of the equation to the other, the order of its points, or lines, being reversed, provided one of its points or lines is found on the other side.

Thus if $$A^{-}B\,B^{-}C = A^{-}D\,D^{-}C$$

and $$(A)^{\frown}(B)\,(B)^{\frown}(C) = (A)^{\frown}(D)\,(D)^{\frown}(C),$$

then $$(A)^{-}(B) = (A)^{-}(D)\,(D)^{-}(C)\,(C)^{-}(B),$$

and $$(A)^{\frown}(B) = (A)^{\frown}(D)\,(D)^{\frown}(C)\,(C)^{\frown}(B).$$

It is evident that of a series of consecutive segments on one side of the equation only the first and last could be thus transferred to the other side.

Thus in the last equations $D^{-}C$ and $(D)^{\frown}(C)$ cannot be transferred, but the first and last segments on the right-hand side may. Thus

$$D^{-}A\,A^{-}B = D^{-}C\,C^{-}B$$

and $$(D)^{\frown}(A)\,(A)^{\frown}(B) = (D)^{\frown}(C)\,(C)^{\frown}(B).$$

In the composition of linear vectors however (with the usual notation),

the equation

$$AB + CD = AD + CB$$

is true if the line $B\delta$, parallel to and equal to CD, is considered its equivalent, and the line $D\delta$, parallel to CD and equal to it, is considered the equivalent of CD.

The equation then means, in contiguous segments,

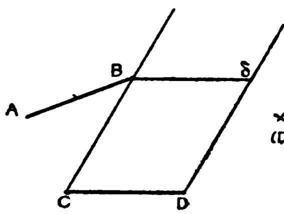
$$AB + B\delta = AD + D\delta,$$

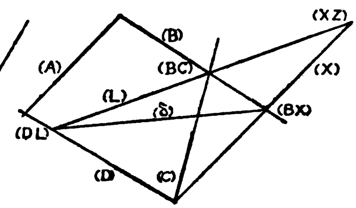
or $$A^{-}B\,B^{-}\delta = A^{-}D\,D^{-}\delta,$$

which the figure shews to be true.

We require therefore to find an analogous 'composition' of angular vectors when the angles 'compounded' are not contiguous.

One form of correspondence may be found thus:—





To compound $A\bar{\ }B$ with $C\bar{\ }D$, or to find an 'equivalent' for $C\bar{\ }D$ starting from B.

Let XZ represent the line at infinity, and call it the 'director line.'

Let X represent the intersection of CD with the 'director line' XZ.

Join BX.

Join BC.

Let L be the point in which CB and XZ intersect.

Let δ be the point in which DL intersects BX.

$B\delta$ is the required equivalent of $C\bar{\ }D$.

To compound $(A)\widehat{\ }(B)$ with $(C)\widehat{\ }(D)$, or to find an equivalent for $(C)\widehat{\ }(D)$ starting from (B).

Let (XZ) be a fixed point, and call it the 'director point.'

Let (X) be the line joining (CD) to the 'director point' (XZ).

Let (BX) be the intersection of (B) and (X), and (BC) the intersection of (B) and (C).

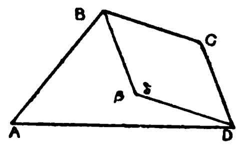
Let (L) be the line joining (BC) to (XZ).

Let (δ) be the line joining (DL) and (BX).

$(B)\widehat{\ }(\delta)$ is the required equivalent of $(C)\widehat{\ }(D)$.

From the nature of the correspondence itself the equation of angular vectors will be subject to the rule of transference of angles with containing lines in reversed order corresponding to the transference of segments with extremities in reversed order.

This may be illustrated in a simple example.



Let an equation of linear vectors be

$$A^{-}B = A^{-}D\,D^{-}C\,C^{-}B.$$

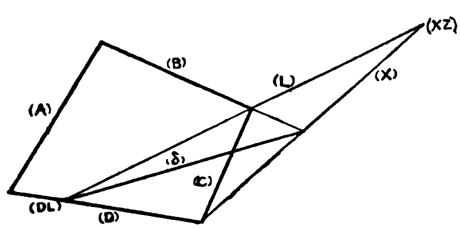
The equation

$$A^{-}B\,C^{-}D = A^{-}D\,C^{-}B$$

is true as meaning

$$A^{-}B\,B^{-}\delta = A^{-}D\,D^{-}\beta,$$

where $B^{-}\delta$ and $D^{-}\beta$ are the equivalents of $C^{-}D$ and $C^{-}B$ respectively, if δ coincides with β; which is obviously true since $B\delta$ is parallel to and equal to CD, and $D^{-}\beta$ equal and parallel to CB.



Let an equation of angular vectors be

$$(A)^{\frown}(B) = (A)^{\frown}(D)\ (D)^{\frown}(C)\ (C)^{\frown}(B).$$

The equation

$$(A)^{\frown}(B)\ (C)^{\frown}(D) = (A)^{\frown}(D)\ (C)^{\frown}(B)$$

is true as meaning

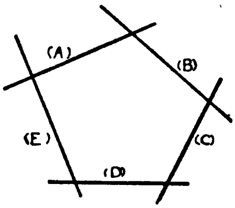
$$(A)^{\frown}(B)\ (B)^{\frown}(\delta) = (A)^{\frown}(D)\ (D)^{\frown}(\beta)$$

(where $(B)^{\frown}(\delta)$ and $(D)^{\frown}(\beta)$ are equivalents of $(C)^{\frown}(D)$ and $(C)^{\frown}(B)$ respectively) if (δ) coincides with (β).

Construct (δ) as above. To construct (β) we have to join (BC) to (XZ). The joining line is (L). The intersection of (D) and (L) is (DL). Join (DC) to (XZ). The joining line is (X). Then (β) is the line joining (BX) to (DL). But this is (δ). Hence the above equation is true.

The figure may be simplified by making the director point the point at infinity upon a given line, e.g. a line of the figure such as (A).

There is, however, a manner of compounding angular vectors as simple as that for linear vectors, and depending upon the same property of parallels.



Let the composition of $(A)\widehat{\ }(B)$ with $(H)\widehat{\ }(K)$ mean that the traverse $(A)\widehat{\ }(B)$ is succeeded by the traverse $(B)\widehat{\ }(k)$ where the angle $(B)\widehat{\ }(k) =$ the angle $(H)\widehat{\ }(K)$; that is, the equivalent of $(H)\widehat{\ }(K)$ is an angle $(B)\widehat{\ }(k)$ which is equal to it. Also let parallel positions be considered equivalent.

Let $(A)\widehat{\ }(B) = (A)\widehat{\ }(E)\ (E)\widehat{\ }(D)\ (D)\widehat{\ }(C)\ (C)\widehat{\ }(B)$.

Transfer $(E)\widehat{\ }(D)$.

The equation

$$(A)\widehat{\ }(B)\ (D)\widehat{\ }(E) = (A)\widehat{\ }(E)\ (D)\widehat{\ }(C)\ (C)\widehat{\ }(B)$$

is equivalent to

$$(A)\widehat{\ }(B)\ (B)\widehat{\ }(\epsilon) = (A)\widehat{\ }(E)\ (E)\widehat{\ }(\gamma)\ (\gamma)\widehat{\ }(\beta),$$

where $(B)\widehat{\ }(\epsilon)$, $(E)\widehat{\ }(\gamma)$ and $(\gamma)\widehat{\ }(\beta)$ are the equivalents of $(D)\widehat{\ }(E)$, $(D)\widehat{\ }(C)$ and $(C)\widehat{\ }(B)$ respectively. And the equation is true if (β) coincides with (ϵ), or is parallel to it.

Now since the angle $(B)\widehat{\ }(\epsilon) = (D)\widehat{\ }(E)$, $(E)\widehat{\ }(\gamma) = (D)\widehat{\ }(C)$ and $(\gamma)\widehat{\ }(\beta) = (C)\widehat{\ }(B)$, the last equation is evidently true if the traversing line beginning with the position (A), revolved through a fixed point, and traversed all the successive angles at that point; for then (β) would coincide with (ϵ).

But the actual traversing line is always parallel to a line thus revolving through a fixed point, and the angles between its successive positions are equal to those between the successive positions of the line which revolves through the fixed point.

Hence (β) will coincide with (ϵ), or be parallel to it, and the equation with the transferred angle is true.

Note upon a reciprocal relation of latus rectum and directrix (§ 75, *page* 143).

The reciprocal relation of latus rectum and directrix, which may be put thus,

If two tangents are drawn to a conic from a point in the $\left\{\begin{array}{l}\textit{latus rectum}\\ \textit{directrix}\end{array}\right\}$ *the intercept of these tangents upon the* $\left\{\begin{array}{l}\textit{directrix}\\ \textit{latus rectum}\end{array}\right\}$ *is bisected by the axis,*

does not appear to be in the books. If it is new it is only an instance of the way in which a new method may lead to a result easy to obtain by usual methods but not directly suggested by them.

It would of course be anticipated that this theorem would be a case of some theorem depending on the harmonic properties of pole and polar, and the writer is indebted to Mr. P. K. Tollit, of Magdalen College, Oxford, Head Master of Derby School, for pointing out that it is a case of the following:—

If from any point in the polar of a point O tangents are drawn to a conic meeting a line through O parallel to the polar of O in the points T and T', then $OT = OT'$.

It is obvious that this follows at once from the harmonic properties of pole and polar.

Oxford
Printed at the Clarendon Press
By Horace Hart, M.A.
Printer to the University